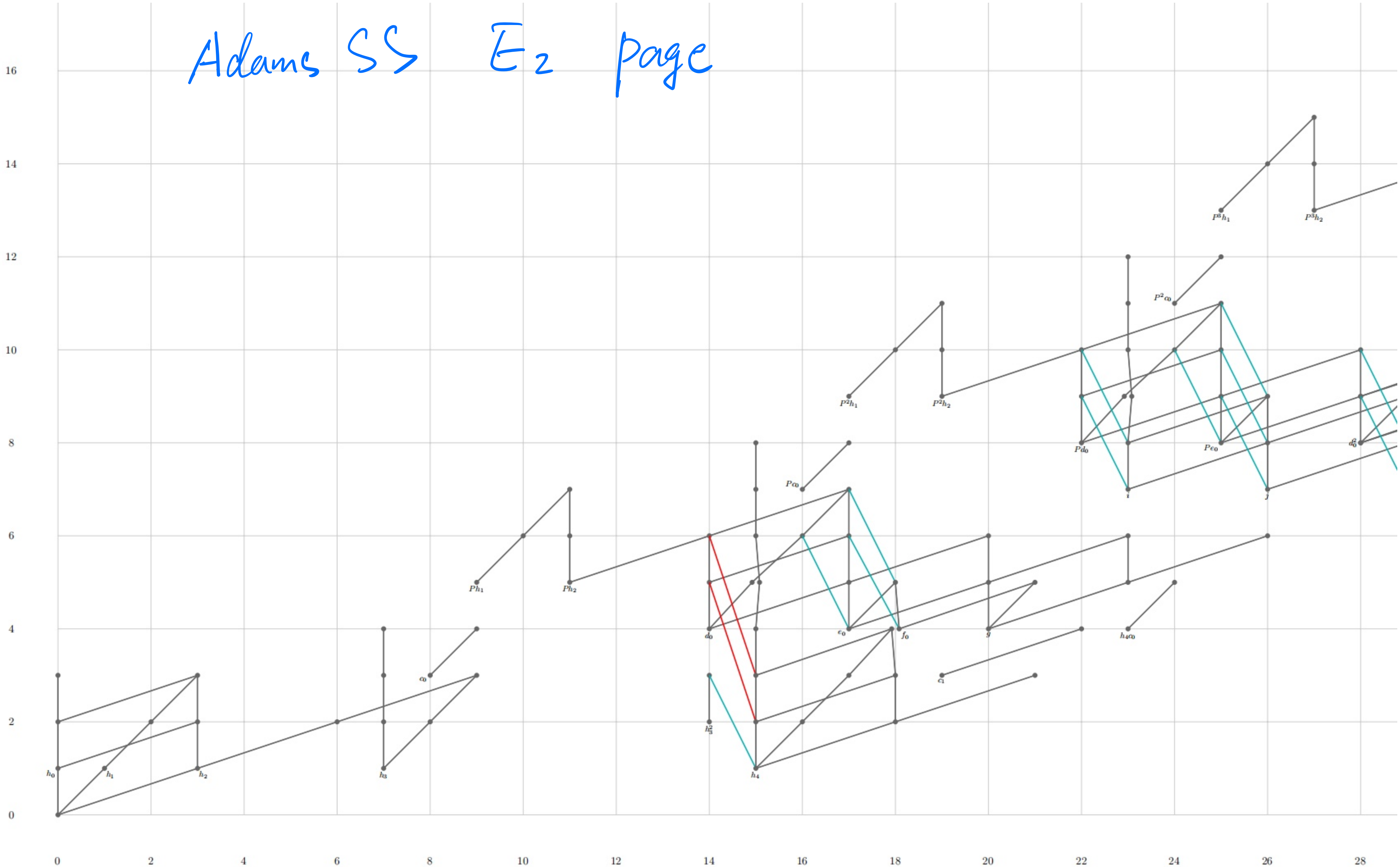
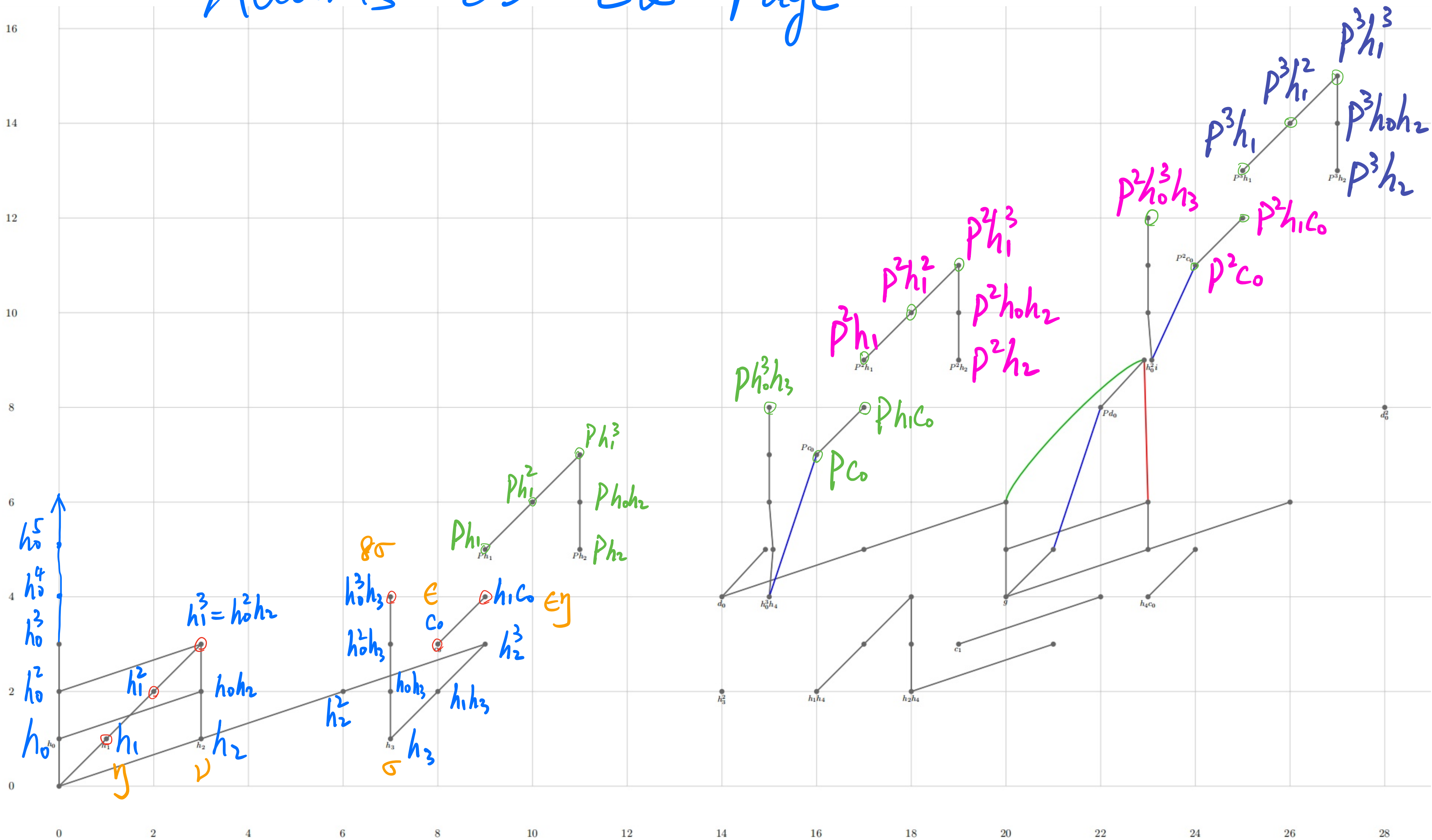
Adams SS E_2 page



Adams SS E_∞ Page



From the Adams's classical v_1-periodicity to the Andrews's motivic w_1-periodicity in homotopy groups

Reference:

1966. Adams. "On the groups J(X) IV" 1966

1966. Adams. "A periodicity theorem in homological algebra"

2018. Andrews. "New families in the homotopy of the motivic sphere spectrum"

Classical Chromatic phenomenon

Self map $\Sigma^d X \xrightarrow{f} X$

f is nilponent if $\exists\, k \in \mathbb{N}$ s.t. $f^k \sim 0$ null homotopy

otherwise, f is periodic

To investigate the homotopy grps of X.

<u>Step 1</u>. find periodic self map $\Sigma^d X \xrightarrow{f} X$

<u>Step 2</u>. Compute $\pi_*(X)[f^{-1}]$ f local part of $\pi_*(X)$

<u>Step 3</u>. Compute f torsion part by replace X by X/f (cofiber of f)

Devinatz - Hopkins - Smith 1988 (Nilpotence thm)

Self map f is <u>nilpotent</u> iff $\overline{MU}_*(f)$ is <u>trivial</u>

A p-local finite complex X has type n if n is the smallest integer such that $\overline{K(n)}_*(X)$ is <u>nontrivial</u>.

If X is contractible it has type ∞.
(If $\overline{K(n)_*(X)}$ vanishes, then so does $\overline{K(n-1)_*(X)}$).

Hopkins. Smith. 1987 (periodicity thm)
X is p-local finite CW-complex of type n.
$\exists$ self map $f : \Sigma^{d+i} X \to X^i$ for some i
s.t. $K(n)_*(f)$ is an isomorphism and
$K(m)_*(f) = 0$ for $m > n$. (If $m < n$. $\overline{K(m)}(X) = 0$ so $\overline{K(m)_*} f = 0$)

$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ $|v_n| = 2(p^n - 1)$.
we call this map a v_n-map. $d = 2(p^n - 1)$.

Example.
(Nishida 1973) $\forall$ element is $\pi_k(S^0)$ is nilpotent if $k > 0$. i.e. the only periodic map represented in $\pi_*(S^0)$ is $p \in \mathbb{Z} \cong \pi_0(S^0)$.

(Adams 1966) ①. Calculate the image of J-homomorphism.
② discover v_1-periodic map on the cofiber $S^0 \xrightarrow{p} S^0$
$p=2$ $\exists$ $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$
$p=3$ $\exists$ $v_1 : \Sigma^{2p-2} S/p \longrightarrow S/p$
which induce isomorphism on $K(1)$, and produce infinite families

$$S^{qn} \xrightarrow{i} \Sigma^{qn} S^0/p \xrightarrow{v_1^n} S^0/p \xrightarrow{c} S^1 \quad \in \pi_{qn-1}(S^0) \quad (*)$$

where i and c are inclusion and collapse
$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} S^0/p \xrightarrow{c} S^1$$

Adams' work gave birth to chromatic homotopy theory at $p=2$. Adams $v_1^4 : \Sigma^8 S^0/2 \to S^0/2$

How to get more infinite families other than the family given in $(*)$. We fix the right part of $(*)$ and lift other maps from $\pi_*(S^0)$ to $\pi_*(S^0/2)$ and replace the first map by this map. i.e. "insert" some iteration of v_1^4.

Suppose we have a map $f \in \pi_m(S^0)$ which can be lift to $\pi_{m+1}(S^0/2)$. Then we can produce elements $P^n(f) \in \pi_{8n+m}(S^0)$

$$\begin{tikzcd}
 & S^0/2 \arrow[loop right, "v_1^{4n}"] \arrow[d, "c"] \\
S^{m+1} \arrow[ur, dashed, "\exists l"] \arrow[r, "\Sigma f"] & S^1
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\Sigma^{8n} S^0/2 \arrow[r, "v_1^{4n}"] & S^0/2 \arrow[d, "c"] \\
\Sigma^{8n} S^{m+1} \arrow[u, "l"] \arrow[r, bend left] & S^1
\end{tikzcd}$$

$$P^n(f) : S^{8n+m} \to S^0 \in \pi_{8n+m}(S^0)$$

For homotopy classes of order 2, the following maps can be lift to the spectrum $S/2$.

$\eta \in \pi_1(S^0)$, $\eta^2 \in \pi_2(S^0)$, $\eta^3 \in \pi_3(S^0)$

$8\sigma \in \pi_7(S^0)$, $\epsilon \in \pi_8(S^0)$, $\epsilon\eta \in \pi_9(S^0)$

Remark:

$P^{m-1}(8\sigma) \in \pi_{8m-1}(S^0)$ is just the infinite family given in $(*)$

Toda showed that the composition

$$S^8 \xrightarrow{i} \Sigma^8 S/2 \xrightarrow{v_1^4} S/2 \xrightarrow{c} S^0 \quad \text{is } 8\sigma \in \pi_7(S^0)$$

So 8σ can be considered as P operation on a null homotopy map $0 \in \pi_{-1}(S^0)$ i.e. $S^0 \xrightarrow{i} S^0/p \xrightarrow{c} S^1$.

Adams Spectral Sequence

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}(S^0)$$

The periodicity phenomenon can be observed on algebraic aspect.

<u>Thm</u> (Adams 1966) $\forall r \geq 2$. $\exists$ a suitable neighbourhood of the line $t = 3s$ (i.e. $s = \frac{1}{2}(t-s)$).
we have periodic isomorphism

$$P_r : \mathrm{Ext}_A^{s,t} \xrightarrow{\cong} \mathrm{Ext}_A^{s+2^r,\, t+3\times 2^r}$$

$$P_r(x) = \langle h_{r+1}, h_0^{2^r}, x \rangle$$

$$P(-) = P_1(-) = \langle h_3, h_0^4, - \rangle = \langle h_0^3 h_3, h_0, - \rangle$$

Motivic story.

Stable category of Schemes over spec $\mathbb{C}$

The homotopy groups have 2 grads, the first carries topology information while the second showcases algebraic information.

$$S^{1,0} \simeq \Delta^1 / \{0,1\}$$

$$S^{1,1} \simeq \mathbb{G}_m \cong \mathrm{Spec}\, \mathbb{C}[T, T^{-1}]$$

$$\pi_{m,n}(S^{0,0}) := [S^{m,n}, S^{0,0}] \qquad S^{m,n} = \Sigma^{m-n}(S^{1,1})^{\wedge n}$$

Voevodsky

$$H\mathbb{F}_{p\,*,*} \cong \mathbb{F}_p[\tau] \qquad |\tau| = (0,-1)$$

Classical	Motivic
Hopf map $\eta: S^3 \to S^2$	$\eta: \mathbb{C}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{CP}^1$
represent a element $\eta \in \pi_1(S^0)$	$(x,y) \longmapsto [x:y]$
η is nilpotent, $\eta^4 = 0$	$\eta \in \pi_{1,1}(S^{0,0})$
	η is periodic

In ASS. h_1 represent y.
$h_1^4 = 0$

In mot ASS. $h_1^4 \neq 0$.
$h_1^k \neq 0$ for $\forall k \geq 1$
While $\tau \cdot h_1^4 = 0$

τ local part of motivic stable homotopy theory
$=$ classical stable homotopy theory

Miller square.

2 most effective methods of computing $\pi_*(S^0)$ are

ASS: $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}(S^0)$

ANSS: $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_{t-s}(S^0)$

Connection between them are:

$$\begin{tikzcd}
 & \mathrm{Ext}_P^{s,t}(\mathbb{F}_2, \mathrm{Ext}_Q^k(\mathbb{F}_2,\mathbb{F}_2)) \cong \mathrm{Ext}_P^{s,t'}(\mathbb{F}_2, I^k/I^{k+1}) \arrow[dl, Rightarrow, "\text{Cartan-Eilenberg SS}"'] \arrow[dr, Rightarrow, "\text{Algebraic Novikov SS}"] & \\
\mathrm{Ext}_A^{s+k,t}(\mathbb{F}_2,\mathbb{F}_2) \arrow[dr, Rightarrow, "\text{Adams SS}"'] & & \mathrm{Ext}^{s,t-k}_{BP_*BP}(BP_*, BP_*) \arrow[ll, "\text{Thom reduction}"'] \arrow[dl, Rightarrow, "\text{Adams-Novikov SS}"] \\
 & \pi_{t-s-k} S^0 &
\end{tikzcd}$$

where

$I = (2, v_1, v_2, \cdots) \subset BP_* \cong \mathbb{Z}_2[v_1, v_2, \cdots]$

$P = \mathbb{F}_2[\xi_1^2, \xi_2^2, \cdots] \subset A = H\mathbb{F}_{2*}H\mathbb{F}_2$

$Q = A \otimes_P \mathbb{F}_2$

Since y corresponding to ξ_1^2. Miller suggested that there may be other non-nilpotent self maps corresponding to $\xi_2^2, \xi_3^2, \cdots \in P$. we call a self map corresponding to ξ_{n+1}^2, w_n.

When $p=2$. $|v_1| = 2$, $|v_2| = 6$
$|v_1^4| = 8$, $|v_2^{32}| = 192$

$|w_1| = (5,3)$, $|w_2| = (13,7)$

$|w_1^4| = (20,12)$, $|w_2^{32}| = (416,224)$

Thm (Andrews 2017)

$\exists \; w_1^4 : \Sigma^{20,12} S/\eta \to S/\eta$ periodic map

It offers 6 infinite families.

$\nu \in \pi_{3,2}(S^{0,0})$, $\nu^2 \in \pi_{6,4}(S^{0,0})$, $\nu^3 \in \pi_{9,6}(S^{0,0})$

$\eta^2\eta_4 \in \pi_{18,11}(S^{0,0})$, $\bar{\sigma} \in \pi_{19,11}(S^{0,0})$, $\bar{\sigma}\nu \in \pi_{22,13}(S^{0,0})$

Let g be the motivic analogue of Adams' periodicity P.
This periodic operation bring some proper homotopy classes in $\pi_{p,q}(S^{0,0})$ to $\pi_{p+20n,\, q+12n}(S^{0,0})$ by iterating n times.
The above 6 classes are suitable candidate.
we have

$g^n(\nu) \in \pi_{3+20n,\, 2+12n}(S^{0,0})$, $g^n(\eta^2\eta_4) \in \pi_{18+20n,\, 11+12n}(S^{0,0})$, etc.

Remark:

(Toda) $S^8 \xrightarrow{i} \Sigma^8 S/2 \xrightarrow{v_1^4} S/2 \xrightarrow{c} S^0$ is $8\sigma = 2^3 \cdot \sigma$

(New find) $S^{18,11} \hookrightarrow \Sigma^{20,12} S/\eta \xrightarrow{w_1^4} S/\eta \xrightarrow{c} S^{0,0}$ is $\eta^2\eta_4$

$\eta^2\eta_4$ can be considered as g operation on a null homotopy map $0 \in \pi_{-2,-1}(S^{0,0})$ i.e. $S^{0,0} \xrightarrow{i} S^{0,0}/\eta \xrightarrow{c} S^{2,1}$.

Sketch of proof:

Step 1 : construct a map, call it w_1^4.

Step 2 : prove it is periodic (non-nilpotent).

Some notation :

$$\begin{tikzcd} S^{1,1} \arrow[r, "\eta"] \arrow[drr] & S^{0,0} \arrow[r, "i"] \arrow[dr] & S/\eta \arrow[r, "c"] \arrow[d] & S^{2,1} \arrow[dl] \\ & & S/\eta & \end{tikzcd} \qquad (**1)$$

Hom spectra

$$\Sigma^{-2,-1} S/\eta \xrightarrow{i = c^*} End(S/\eta) \xrightarrow{c = i^*} S/\eta \xrightarrow{\eta^*} \Sigma^{-1,-1} S/\eta \qquad (**2)$$

The detection map. (motivic version Thom reduction)

$$D : Ext_{BP_{**}BP}(BP_{**}, BP_{**}(-)) \longrightarrow Ext_{\underset{HF_{2**}HF_2}{\overset{\shortparallel}{A}}}(\mathbb{F}_2, \mathbb{F}_2(-))$$

In $(-)$ put $S^{0,0}$ and S^0

$S^{0,0}/\eta$ $\quad S^0/2$

$End(S^{0,0}/\eta)$ $\quad End(S^0/2)$

$BP_{**} = \mathbb{Z}_2[\tau, v_1, v_2, \cdots]$ $\qquad A = \mathbb{F}_2[\xi_1, \xi_2, \cdots\cdots]$

$|\tau| = (0,-1)$ $\quad |v_n| = (2^{n+1}-2, 2^n-1)$ $\qquad |\xi_n| = 2^n - 1$

$BP_{**}BP = BP_{**}[t_1, t_2, \cdots]$ $\qquad |t_n| = |v_n|$

$$D: \quad \begin{aligned} 1 &\longmapsto 1 \\ \tau &\longmapsto 0 \\ v_n &\longmapsto 0 \\ t_1 &\longmapsto \xi_1 \end{aligned}$$

η, ν, σ in $\pi_{*,*}(S^{0,0})$ are detected by $\alpha_1, \alpha_{2/2}, \alpha_{4/4}$ in mot ANSS

And in mot ANSS, $\beta_{4/3}$ detects a homotopy class $\eta_4 \in \pi_{16,9}(S^{0,0})$

<u>Thm</u>. $D(\overset{\eta}{\alpha_1}) = h_0^{\,2}$, $D(\overset{\nu}{\alpha_{2/2}}) = h_1^{\,\eta}$, $D(\overset{\sigma}{\alpha_{4/4}}) = h_2^{\,\nu}$

$D(\overset{\eta_4}{\beta_{4/3}}) = h_0 h_3^{\,2\sigma}$, $D(\overset{\bar{\sigma}}{z_{19}}) = c_0^{\,\epsilon}$

Lemma. In motivic ANSS,

$\alpha_1^3 \beta_{4/3} = 0$ and $\alpha_1^2 \beta_{4/3} \neq 0$

Step 1. Existence of map $w_1^4 : \Sigma^{20,12} S/y \to S/y$

Apply derived functor on sequence $(**1)\&(**2)$, we get

$$\mathrm{Ext}^{4,24,12}_{BP_{**}BP}(BP_{**}, BP_{**}(\mathrm{End}(S/y))) \longrightarrow \mathrm{Ext}^{4,24,12}_{BP_{**}BP}(BP_{**}, BP_{**}(S/y))$$

$$\exists x \longmapsto \exists y$$

$$\longrightarrow \mathrm{Ext}^{4,22,11}_{BP_{**}BP}(BP_{**}BP, BP_*(S^{0,0}))$$

$$\longmapsto \alpha_1^2 \beta_{4/3}$$

There exists a preimage of $\alpha_1^2 \beta_{4/3}$. we call it y. and there exists a pre image of y, we call it x.

$$\underbrace{S^{1,1} \xrightarrow{y} S^{0,0} \xrightarrow{i} S/y \xrightarrow{c} S^{2,1} \xrightarrow{\Sigma y} S^{1,0}}$$

(y is represented by α_1)

$$H_*(S/y) \xrightarrow{H_*(C)} H_*(S^{2,1}) \xrightarrow{H_*(\Sigma y)} H_*(S^{1,0})$$

$$\exists y \longmapsto \alpha_1^2 \beta_{4/3} \xmapsto{\cdot \alpha_1} 0 = \alpha_1^3 \beta_{4/3}$$

similarly

$$\Sigma^{-2,-1} S/y \xrightarrow{i=c^*} \mathrm{End}(S/y) \xrightarrow{c=i^*} S/y \xrightarrow{y^*} \Sigma^{-1,-1} S/y$$

$$H_*(\mathrm{End}(S/y)) \longrightarrow H_*(S/y) \xrightarrow{H_*(y^*)} H_*(\Sigma^{-1,-1} S/y)$$

$$\exists x \longmapsto y \xmapsto{\cdot \alpha} 0$$

x is a permanent cycle in mot AASS

for degree reason, all differential is trivial.

Remark: The composition $c \circ w_1^4 \circ i = y^2 y_4 \sim \alpha_1^2 \beta_{4/3}$

$$S^{18,11} \xrightarrow{i} \Sigma^{18,11} S/y \xrightarrow{w_1^4} \Sigma^{-2,-1} S/y \xrightarrow{c} S^{0,0}$$

Comparision $c \circ v_1^4 \circ i = 8\sigma = 2^3 \sigma \sim h_0^3 h_3$

$$D(\alpha_1^2 \beta_{4/3}) = h_0^2 \cdot h_0 h_3 = h_0^3 h_3$$

Step 2. w_1^4 is non-nilpotent.

Adams 66. $\exists\ v_1^4 : \Sigma^8 S^0/2 \to S^0/2$

In our language. element $\underset{\sim 8\sigma}{h_0^3 h_3}$ can be lift to $\bar{x}$

$$\mathrm{Ext}_A^{4,12}(\mathbb{F}_2, H_*(\mathrm{End}\, S/2)) \xrightarrow{c = i^*} \mathrm{Ext}_A^{4,12}(\mathbb{F}_2, H_*(S/2)) \xrightarrow{c} \mathrm{Ext}_A^{4,11}(\mathbb{F}_2, H_*(S^0))$$

$$\exists\ \bar{x} \longmapsto h_0^3 h_3 \sim 8\sigma \quad 2^3\sigma$$

There is a commutative diagram.

$$\begin{tikzcd}
\text{mot ANSS} & \mathrm{Ext}^{4,24,12}_{BP_{**}BP}(\mathrm{End}(S/y)) \arrow[r, "\exists \text{ lift}"] \arrow[d, "D"] & \mathrm{Ext}^{4,22,11}_{BP_{**}BP}(S^{0,0}) \arrow[d, "D"] \\
\text{ASS} & \mathrm{Ext}_A^{4,12}(\mathrm{End}(S/2)) \arrow[r, "\exists \text{ lift}"] & \mathrm{Ext}_A^{4,11}(S^0)
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_1^2 \beta_{4/3} \sim y^2 y_4 \arrow[d, mapsto] \\
\bar{x} \arrow[r, mapsto] & h_0^3 h_3 \sim 8\sigma
\end{tikzcd}$$

Since $\bar{x}$ is non-nilpotent, $D(x) = \bar{x}$.

$\bar{x}^n \neq 0$ for $\forall\ n \geq 1$, $D(x^n) = D(x)^n = \bar{x}^n \neq 0$

$\Rightarrow x^n \neq 0$ for $\forall\ n \geq 1$. x is non-nilpotent, Thm ✓