FORMAL GROUPS AND PIPE FORMAL GROUPS

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Abstract. We prove theorem 2.1 using the method of proof by way of contradiction. This theorem states that for any set A, that in fact the empty set is a subset of A, that is $\emptyset \subset A$.

1. Formal schemes

In algebraic geometry, a formal scheme is used to detect the local behavior around a closed point. For example, Let $R = k[x]$ for some field k. The maximal ideal (x) corresponds to the closed point $[0]$. To study the local behavior around this closed point, one has a sequence

$$\mathrm{Spec}(k) \cong \mathrm{Spec}(k)[x]/x \to \mathrm{Spec}(k)[x]/x^2 \to \cdots \to \mathrm{Spec}(k)[x]/x^n \to \cdots .$$

In each stage, Spec $k[x]/x^n$ has the underlying space $[0]$, but the functions are more. This indicates us to take the colimit of this sequence. Unfortunately, the category **Sch** of schemes does not have all limits and colimits.

Remark 1. *The category of locally ringed spaces has all limits and colimits. The category* **Aff** *of affine schemes is the opposite category of* **Ring**. *We have the adjunction*

$$\Gamma : \mathbf{Sch} \leftrightarrows \mathbf{Ring}^{op} : \mathrm{Spec}$$

with Spec *being a right adjoint. Hence it preserves the limits in* $\mathbf{Ring}^{op}$, *or equivalently, colimits in* **Ring**.

No matter in what cases, there is no evidence that the colimit should exist. Hence we have the following definition. Now we say a scheme means an affine scheme in all of the notes, and denote the category of affine scheme by $\mathfrak{X}$, the full subcategory of $\mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$ consisting of representable functors.

Definition 1. *A formal scheme X is a small filtered colimit of scheme X_i. As we already explained, this colimit may not exist in $\mathfrak{X}$. We can embed $\mathfrak{X}$ into* $\mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$, *where the later always has colimits, pointwisely.*

To be more concrete, for each ring R, we have

$$X(R) = \mathrm{colim} X_i(R).$$

Definition 2. *Let $X = \mathrm{colim} X_i$ and $Y = \mathrm{Y_j}$ be formal schemes. Define*

$$\widehat{\mathfrak{X}}(X, Y) = \lim_i \operatorname*{colim}_j \mathfrak{X}(X_i, Y_j).$$

We denote $\widehat{\mathfrak{X}}(X, \mathbb{A}^1)$ by $\mathcal{O}_X$, where $\mathbb{A}^1 = \mathbf{Ring}(\mathbb{Z}[t], -)$. To be precise,

$$\mathcal{O}_X = \lim \mathcal{O}_{X_i}.$$

Remark 2. *From the definition, $\widehat{\mathfrak{X}}$ is actually the same as $\mathcal{I}nd(\mathfrak{X})$. Note that in general, one has*

$$[\mathrm{colim}\, X_i, Y] = \lim [X_i, Y].$$

By the definition of $\operatorname{colim} Y_j$*, we have*

$$\widehat{\mathfrak{X}}(X,Y) = \lim_i \widehat{\mathfrak{X}}(X_i, Y) = \lim_i \operatorname*{colim}_j \mathfrak{X}(X_i, Y_j).$$

This is how we define morphisms in $\widehat{\mathfrak{X}}$*.*

Example 1. *Let* $N_i = \operatorname{Spec} \mathbb{Z}[x]/x^n$*. The resulting formal scheme is denoted by* $\hat{\mathbb{A}}^1$*. Note that* $\hat{\mathbb{A}}^1(R) = \operatorname{colim}\mathbf{Ring}(\mathbb{Z}[x]/x^n, R)) = \operatorname{Nil}(R)$*. And* $\mathcal{O}_{\hat{\mathbb{A}}^1} = \mathbb{Z}[\![x]\!]$*.*

The category $\widehat{\mathfrak{X}}$ has better categorical properties than $\mathfrak{X}$.

1. $\widehat{\mathfrak{X}}$ has all small colimits and finite limits.
2. finite limits commute with small colimits in $\widehat{\mathfrak{X}}$.

There are a special kind of formal schemes, called *solid* formal scheme, $\operatorname{Spf}(R)$, which we will define right now.

Definition 3. *A linear topologized ring is a ring* R *equipped with a neighborhood system around* 0 *consisting of open ideals, which forms a topological basis under translation. The category of such rings and continuous maps is denoted by* **LRing**.

We can equip any ring S with discrete topology, which yields a fully faithful embedding $\mathbf{Ring} \to \mathbf{LRing}$. Suppose $R \in \mathbf{LRing}$, $S \in \mathbf{Ring}$, f is a continuous map from R to S. We must have $f^{-1}(0) = J$ an open ideal in R. Hence f is equivalent to a map $R/J \to S$ between rings. All open ideals in R form a cofiltered system under inclusion maps. Hence we have

$$\mathbf{LRing}(R,S) = \operatorname*{colim}_J \mathbf{Ring}(R/J, S).$$

Therefore we define $\operatorname{Spf}(()R) \in \mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$ by

$$\operatorname{Spf}(R)(S) = \operatorname*{colim}_J \mathbf{Ring}(R/J, S).$$

Definition 4. *A solid formal scheme is a formal scheme which is isomorphic to* $\operatorname{Spf}(R)$ *for some linearly topologized ring* R*. The solid formal schemes form a full subcategory* $\widehat{\mathfrak{X}}_{sol}$ *of* $\widehat{\mathfrak{X}}$*.*

Given a linearly topologized ring R, we have the related cofiltered system $\{R/J\}$, where J runs through all open ideals. The limit of this system is denoted by $\widehat{R}$, called the completion of R. The ring $\widehat{R}$ automatically inherits a topological structure from R. The preimage $\overline{J}$ of J under the natural map $\widehat{R} \to R/J$ forms a neighborhood system around 0 in $\widehat{R}$. It is easy to check $\operatorname{Spf}(\widehat{R}) = \operatorname{Spf}(R)$. A ring R is complete or a formal ring if $R = \widehat{R}$. The category of formal rings is denoted by **FRing**, which is a full subcategory of **LRing**.

Note that

$$\widehat{\mathfrak{X}}(X, \operatorname{Spf}(R)) = \lim_i \widehat{\mathfrak{X}}(X_i, \operatorname{Spf}(R)) = \lim_i \mathbf{LRing}(R, \mathcal{O}_{X_i}) = \mathbf{LRing}(R, \mathcal{O}_X).$$

Hence we have the adjoint pairs:

$$\mathcal{O} : \widehat{\mathfrak{X}} \leftrightarrows \mathbf{LRing}^{op} : \operatorname{Spf}.$$

We have the unit map $X \to \operatorname{Spf}(\mathcal{O}_X)$, and the counit $R \to \widehat{R}$, which is just the completion.

Proposition 1. *We have the following propositions.*

1. X *is a solid fomal scheme then* $\mathcal{O}_X$ *is a formal ring.*

(2) *X is solid iff $X \to \mathrm{Spf}(\mathcal{O}_X)$ is an isomorphism.*
(3) *The inclusion functor $\widehat{\mathfrak{X}}_{sol} \to \widehat{\mathfrak{X}}$ is right adjoint to $X \to \mathrm{Spf}(\mathcal{O}_X)$.*
(4) *The inclusion functor $\mathbf{FRing} \to \mathbf{LRing}$ is right adjoint to taking completion.*

Proof. (1) Obvious.
(2) X is solid, then X is isomorphic to $\mathrm{Spf}(R)$ for some R. Therefore $\mathcal{O}_X$ is isomorphic to $\widehat{R}$, which yields the conclusion. The converse is obvious.
(3) The functor $\mathbf{FRing}^{op} \to \widehat{\mathfrak{X}}_{sol}$ sending R to $\mathrm{Spf}(R)$ is fully faithful. Suppose R, S are two formal rings, then

$$\widehat{\mathfrak{X}}_{sol}(\mathrm{Spf}(R), \mathrm{Spf}(S)) = \lim_{J} \widehat{\mathfrak{X}}_{sol}(\mathrm{Spec}(R/J), \mathrm{Spf}(S)) = \lim \mathbf{LRing}(S, R/J) = \mathbf{FRing}(S, R).$$

Therefore by (2), this functor defines an equivalence. The equation

$$\widehat{\mathfrak{X}}(X, \mathrm{Spf}(R)) = \mathbf{LRing}(R, \mathcal{O}_X) = \widehat{\mathfrak{X}}_{sol}(\mathrm{Spf}(\mathcal{O}_X), \mathrm{Spf}(R))$$

implies the inclusion functor being right adjoint to $X \to \mathrm{Spf}(\mathcal{O}_X)$.
(4) The same argument holds.

$$\mathbf{LRing}(R, \widehat{S}) = \mathrm{Spf}(R)(\widehat{S}) = \mathrm{Spf}(\widehat{R})(\widehat{S}) = \mathbf{FRing}(\widehat{R}, \widehat{S}).$$

□

Definition 5. *Suppose $R \to S$ and $R \to T$ are continuous maps in $\mathbf{LRing}$. Define their tensor products $S \otimes_R T$ to be the usual tensor product equipped with linear topology spanned by $S \otimes_R I + J \otimes_R T$, for open ideals $I \subset T$ and $J \subset S$. This actually the pushout in $\mathbf{LRing}$. If both of them are formal rings, then we define $S \widehat{\otimes}_R T$ to be the completion of $S \otimes_R T$, which corresponds to the pushout in $\mathbf{FRing}$ for the completion being a left adjoint.*

2. Formal Groups

Definition 6. *A formal group G over a formal scheme X is a group object in $\widehat{\mathfrak{X}}_X$. We also require that G is isomorphic to $X \times \widehat{\mathbb{A}}^1$ in $\widehat{\mathfrak{X}}_X$. A map $u : G \to \widehat{\mathbb{A}}^1$ makes G isomorphic to $X \times \widehat{\mathbb{A}}^1$ is called a coordinate on G.*

Suppose X is solid. Then $X \times \widehat{\mathbb{A}}^1$ is again solid. From the equivalence of categories, we have $X \times \widehat{\mathbb{A}}^1$ is isomorphic to the Spf of coproduct of $\mathcal{O}_X$ and $\mathbb{Z}[\![t]\!]$ in $\mathbf{FRing}$, which is the completion of $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[\![t]\!] \cong \mathcal{O}_X[\![t]\!]$. Therefore G is solid as well with $\mathcal{O}_G \cong \widehat{\mathcal{O}_X[\![t]\!]}$.

Moreover, if we further assume X is just a scheme, then

$$\mathcal{O}_G \cong \mathcal{O}_X \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}[\![t]\!] \cong \mathcal{O}_X[\![t]\!]$$

for now $\mathcal{O}_X$ is equipped with the discrete topology. A coordinate on G is the same as an isomorphism from G to $X \times \widehat{\mathbb{A}}^1$, which corresponds to a continuous map

$$u : \mathbb{Z}[\![t]\!] \to \mathcal{O}_G$$

which induces an isomorphism

$$\mathcal{O}_X[\![t]\!] \to \mathcal{O}_G.$$

Now since G is a group object, we have a map $G \times_X G \xrightarrow{\mu} G$, which corresponds to

$$\mathcal{O}_G \to \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G$$

of $\mathcal{O}_X$ modules. We also call the latter map μ, and it satisfies following properties.

Identity: There is a map $X \xrightarrow{e} G$ such that the composite

$$X \to G \to X$$

is identity. Moreover we require the composition

$$G \cong X \times_X G \xrightarrow{e \times id} G \times_X G \xrightarrow{\mu} G$$

equals the identity from G to itself.

Equivalently, there is a map $e : \mathcal{O}_G \to \mathcal{O}_X$, such that

$$\mathcal{O}_X \to \mathcal{O}_G \to \mathcal{O}_X$$

is identity and

$$\mathcal{O}_G \xrightarrow{\mu} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \xrightarrow{e \otimes id} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_G \cong \mathcal{O}_G$$

is identity.

Associativity:

$$\begin{array}{ccc} G \times_X G \times_X G & \xrightarrow{\mu \times id} & G \times_X G \\ \downarrow{\scriptstyle id \times \mu} & & \downarrow{\scriptstyle \mu} \\ G \times_X G & \xrightarrow{\mu} & G \end{array} \qquad \begin{array}{ccc} \mathcal{O}_G & \xrightarrow{\mu} & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle id \otimes \mu} \\ \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G & \xrightarrow{\mu \otimes id} & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G. \end{array}$$

Commutativity:

$$\begin{array}{ccc} G \times_X G & \xrightarrow{T} & G \times_X G \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & G & \end{array} \qquad \begin{array}{ccc} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G & \xrightarrow{T} & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & \mathcal{O}_G & \end{array}$$

where T is transposition.

If we choose a coordinate on G, then we have an isomorphism from $\mathcal{O}_G$ to $\mathcal{O}_X[\![t]\!]$. The map μ now is

$$\mathcal{O}_X[\![t]\!] \longrightarrow \mathcal{O}_X[\![x, y]\!]$$

between $\mathcal{O}_X$ modules, which is determined by $f(x, y) = \mu(t) \in \mathcal{O}_X[\![x, y]\!]$. Such power series $f(x, y)$ is called a formal group law over $\mathcal{O}_X$, which satisfies

- $f(0, y) = y$;
- $f(x, f(y, z)) = f(f(x, y), z)$;
- $f(x, y) = f(y, x)$.

Remark 3. *The identity $\mathcal{O}_X[\![t]\!] \to \mathcal{O}_X$ can only be $t \mapsto 0$. Since this map is continuous between $\mathcal{O}_X$ modules, hence is determined by the image of t which is a nilpotent element n in $\mathcal{O}_X$. Note that*

$$f(x, y) = \sum f_i(x) y^i = \sum f_i(y) x^i$$

by commutativity. Hence we have

$$f(n, y) = \sum f_i(n) y^i = y$$

which implies that $f_0(n) = 0$. Hence $f_0(x)$ is divided by x^k for some k. So does $f_0(y)$, that is

$$f(x, y) = f_0(x) + f_0(y) + \cdots .$$

Hence k must be 1 and $n = 0$.

Example 2. *The additive formal group $\mathbb{G}_a = \mathrm{Spf}(\mathbb{Z}[\![t]\!])$ is a formal group over $\mathbb{Z}$. For any ring R, $\mathbb{G}_a(R) \to \mathrm{Spec}(\mathbb{Z})(R)$ is given by the inclusion of rings $\mathrm{Nil}(R) \to *$. The group structure on $\mathbb{G}_a$ is given by*

$$\mathrm{Nil}(R) \times \mathrm{Nil}(R) \to \mathrm{Nil}(R),\ (a, b) \mapsto a + b.$$

If we choose a coordinate $id : \mathbb{Z}[\![t]\!] \to \mathbb{Z}[\![t]\!]$, the nwe get a formal group law $f(x, y) = x + y$.

The multiplicative formal group $\mathbb{G}_m$ over $\mathbb{Z}$ has the same underlying formal scheme. The group structure is given by

$$\mathrm{Nil}(R) \times \mathrm{Nil}(R) \to \mathrm{Nil}(R),\ (a, b) \mapsto a + b + ab.$$

Use the same coordinate, we have a formal group law $f(x, y) = (1 + x)(1 + y) - 1$.

A morphism between two formal groups $\mathbb{G}$ over X and $\mathbb{H}$ over Y is just a commutative diagram in $\widehat{\mathfrak{X}}$ which respects the group structures of $\mathbb{G}$ and $\mathbb{H}$.

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{q} & \mathbb{H} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{H}} & \xrightarrow{q^*} & \mathcal{O}_{\mathbb{G}} \end{array}$$

Let x, y be coordinates on $\mathbb{G}$ and $\mathbb{H}$, then we have isomorphisms $\mathcal{O}_{\mathbb{G}} \cong \mathcal{O}_X[\![x]\!]$ and $\mathcal{O}_{\mathbb{H}} \cong \mathcal{O}_Y[\![y]\!]$ respectively. The morphism q^* sending y to a series $f(x) \in \mathcal{O}_X[\![x]\!]$, which satisfies

$$f(x +_{\mathbb{G}} x') = f(x) +_{\mathbb{H}} f(x').$$

Such series is called a homomorphism between formal group laws.

Example 3. *A crucial endomorphism from $\mathbb{G}$ to itself is multiplication by p. It is induced by*

$$[p] : \mathbb{G} \xrightarrow{\Delta} \underbrace{G \times_X \cdots \times_X G}_{p \ times} \xrightarrow{\mu} G.$$

where Δ is the diagonal. Choose a coordinate, we have $[p](x) = x +_{\mathbb{G}} \cdots +_{\mathbb{G}} x$,

Suppose now X is over $\mathrm{Spec}(\mathbb{F}_p)$ and $q : \mathbb{G} \to \mathbb{H}$ over X with x, y are coordinates on them. Then there must be $a \neq 0 \in \mathcal{O}_X$ and r such that

$$q^*(y) = ax^r \bmod x^{r+1}.$$

Since q is a homomorphism, we have

$$a(x_0^r + x_1^r) = a(x_0 + x_1)^r \bmod (x_0, x_1)^{r+1}.$$

Let $r = p^n m$, we have

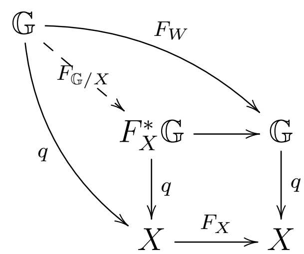
$$x_0^r + x_1^r = (x_0^{p^n} + x_1^{p^n})^m = x_0^r + m x_0^{r-p^n} x_1^{p^n} + \cdots \bmod (x_0, x_1)^{r+1}.$$

Hence m must be 1 and $r = p^n$ is a power of p.

Definition 7. *We call such n the strict height of q. We also let* $\mathrm{Height}(q)$ *to be the strict height of* $\tilde{q} : \mathbb{G}_0 \to \mathbb{H}_0$ *over the special fiber. Finally, we define* $\mathrm{Height}(\mathbb{G})$ *to be* $\mathrm{Height}([p] : \mathbb{G} \to \mathbb{G})$.

Remark 4. *Strict height is always not greater then height obviously. Moreover, we have* $q^*(y) = g(x^{p^n})$*. This is because* $q^*(y) = f(x)$ *must have no constant term due to the continuity. If* $f'(0) \neq 0$*, which means* $f(x) = x + \cdots$*, already meets the requirement. If* $f'(0) = 0$*, then the group law will force* $f'(x) = 0$*, which implies* $f(x) = g(x^p)$*.*

There is a geometric way to think of the strict height of a morphism $f : \mathbb{G} \to \mathbb{H}$ over X. Since X is over $\mathrm{Spec}(\mathbb{F}_p)$, we have a Frobenius map $F_X : X \to X$. The pullback $F_X^*\mathbb{G}$ is also a formal group. If we choose a coordinate x on $\mathbb{G}$ and the induced coordinate y on $F_X^*\mathbb{G}$, then the formal group law on $F_X^*\mathbb{G}$ is given by $g^{(p)}(y, y')$, where g is the formal group law of $\mathbb{G}$ under the coordinate x and $g^{(p)}$ is the series obtained from replacing coefficients g_{ij} in g by g_{ij}^p.

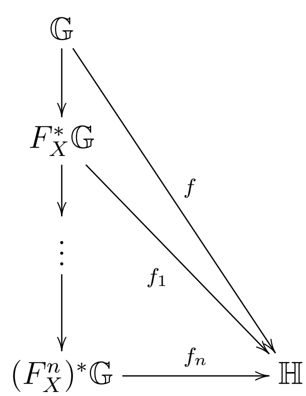


The commutativity of Frobenius maps induces a map $F_{\mathbb{G}/X} : \mathbb{G} \to F_X^*\mathbb{G}$, which is also a group homomorphism. Using the coordinates above, we have $F_{\mathbb{G}/X}^*(y) = x^p$.

Now suppose $f : \mathbb{G} \to \mathbb{H}$ is a group homomorphism with $f^*(y) = g(x^p)$ where x, y are coordinates on $\mathbb{G}$ and $\mathbb{H}$ respectively. From the expression of $f^*(y)$, we know that f factors through

$$\mathbb{G} \xrightarrow{F_{\mathbb{G}/X}} F_X^*\mathbb{G} \to \mathbb{H}.$$

The strict height of f corresponds to the height of the tower:



Proposition 2. *Let* $f : \mathbb{G} \to \mathbb{H}$ *be a nonzero homomorphism over X with* $\mathrm{Height}(\mathbb{G})$ *finite. Then* $\mathrm{Height}(\mathbb{G}) = \mathrm{Height}(\mathbb{H})$ *and* $\mathrm{Height}(f)$ *is finite.*

Proof. Just a direct computation. □

3. Subschemes and Subgroups

Definition 8. *A map $f : X \to Y$ is a closed inclusion, if f is a regular monomorphism, i.e. it is the equalizer of $Y \to Y \coprod_X Y$ in $\widehat{\mathfrak{X}}$. A formal scheme X is a closed subscheme of Y if $f : X \to Y$ is a closed inclusion and X is a subfunctor of Y.*

Remark 5. *$Y \coprod_X Y$ is the pushout via $f : X \to Y$. In the category* **Top***, a regular monomorphism is equivalent to an embedding.*

Example 4. *The map $f : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ induced by $f^* : A \to A/I$ is a closed subscheme. First $\mathrm{Spec}(A/I)(R)$ is the set of all maps from A to R which vanish on I, naturally a subset of $\mathrm{Spec}(A)(R)$. The equalizer of $\mathrm{Spec}(A) \rightrightarrows \mathrm{Spec}(A \times_{A/I} A)$ corresponds to the coequalizer of $A \times_{A/I} A \rightrightarrows A$, which is just A/I.*

Conversely, Suppose $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a closed inclusion. Then f is the equalizer of $\mathrm{Spec}(B) \rightrightarrows \mathrm{Spec}(C)$ which is equivalent to the spectrum of the coequalizer of $C \rightrightarrows B$, which is of the form B/I.

Example 5. *Suppose Y is a closed subscheme of X, i.e. $Y = \mathrm{Spec}(A/I)$ and $X = \mathrm{Spec}(A)$. The same argument implies that $X^\wedge_Y = \mathrm{Spf}(A^\wedge_I) \to X = \mathrm{Spec}(A)$ is a closed inclusion.*

Proposition 3. *If X is a formal scheme, Y is a scheme, then $f : X \to Y$ is a closed inclusion iff there are closed subschemes Y_j such that $X = \mathrm{colim}_j Y_j$.*

Proof. Suppose $X = \mathrm{colim}_j Y_j$, then from previous example, one has f is a closed inclusion.

Conversely, take a presentation $X = \mathrm{colim}_i X_i$, it is easy to verify that the canonical map $X_j \to X$ is a closed inclusion. Hence the composite $X_j \to Y$ is a closed inclusion, which implies X_j is a closed subscheme of Y. □

To investigate in general a closed inclusion $f : X \to Y$ between formal schemes, we need an understanding between categories $\widehat{\mathfrak{X}}_X$ and $\widehat{\mathfrak{X}}_{X_i}$.

Suppose $\{X_i\} : \mathcal{J} \to \widehat{\mathfrak{X}}$ is the filtered diagram with colimit X. Let $D_{\{X_i\}}$ be the category $\mathbf{Fun}(\mathcal{J}, \widehat{\mathfrak{X}})/\{X_i\}$ and $\widehat{\mathfrak{X}}_{\{X_i\}}$ be the full subcategory of $D_{\{X_i\}}$ with each such diagram a pullback for all arrows $u : i \to j$ in $\mathcal{J}$.

$$\begin{array}{ccc} Y_i & \xrightarrow{Y_u} & Y_j \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{X_u} & X_j \end{array}$$

Clearly, we have a functor $F : D_{\{X_i\}} \to \widehat{\mathfrak{X}}_X$ defined by taking colimit, and $G : \widehat{\mathfrak{X}}_X \to D_{\{X_i\}}$ via pullback.

Proposition 4. *The functor F is left adjoint to G, and the functor G is full and faithful. Moreover, G gives an equivalence between $\widehat{\mathfrak{X}}_X$ and $\widehat{\mathfrak{X}}_{\{X_i\}}$.*

Proof. A map from $F\{Y_i\}$ to Z over X is the same as a compatible system of maps $\{Y_i \to Z\}$ over X. Each $Y_i \to X$ factors through X_i, therefore such a map is equivalent to $Y_i \to X_i \times_X Z$ over X_i and the system of such maps is just a map $\{Y_i\} \to G(Z)$ in $D_{\{X_i\}}$, which implies the left adjointness.

Note that $FG(Y) = \operatorname{colim}_{\mathcal{J}} X_i \times_X Y$ and filtered colimits commutes with finite limits. Hence $\operatorname{colim}_{\mathcal{J}} X_i \times_X Y = X \times_X Y = Y$. The fully faithfulness of G follows, as

$$D_{\{X_i\}}(GX, GY) = \widehat{\mathfrak{X}}_X(FGX, Y) = \widehat{\mathfrak{X}}(X, Y).$$

The equivalence of categories follows from intuition but requires some work. □

According to this proposition, suppose $F : X \to Y$ is a closed inclusion, with Y solid. Then f is equivalent to an element in $\widehat{\mathfrak{X}}_{\{Y_i\}}$ with each $X_i \to Y_i$ a closed inclusion, which yields that each X_i is solid. Hence $X = \operatorname{colim} X_i$ is again solid. This proves the following conclusion.

Proposition 5. *A closed subscheme of a solid formal scheme is again solid.*

Proof. This follows from proposition 4. Moreover, suppose X is a solid formal subscheme of solid formal subscheme Y, then we have $\mathcal{O}_X = \mathcal{O}_Y / J$ for some ideal J. □

Definition 9. *Let C be a formal curve over X, i.e. $C \cong X \times \widehat{\mathbb{A}}^1$ and D is a closed subscheme of C. Suppose X is a scheme, we say D is a divisor of degree n if D is also a scheme and $\mathcal{O}_D$ a free module of rank n over $\mathcal{O}_X$. For general X, we say D is a divisor if for all scheme X', $D \times_X X'$ is a divisor of $C \times_X X'$.*

Now suppose X is a scheme with $X = \operatorname{Spec}(R)$. Choose a coordinate x on C, we have $C \cong \operatorname{Spf}(R[\![x]\!])$. Since D is a divisor of C, we have $D = \operatorname{Spec}(R[\![x]\!]/J)$ for some ideal J such that $x^k \in J$ for some k. Let $\lambda(x)$ be the R-endomorphism of $\mathcal{O}_D = R[\![x]\!]/J$ given by multiplying x and let $f_D(t)$ denote its characteristic polynomial.

Suppose D is of degree n, then $f_D(t)$ is a degree n monic polynomial. By Cayley-Hamilton, $f_D(x) = 0 \in \mathcal{O}_D$, which implies $f_D(x) \in J$. While $R[\![x]\!]/f_D(x)$ is also a free module of rank n over R and $\mathcal{O}_D$ is a quotient of it, which is also free of rank n. Hence $\mathcal{O}_D = R[\![x]\!]/f_D(x)$.

Moreover, since $x^k \in J$, if R is a field, we have $f_D(t) = t^n$. If R is not a field, by passing to the fraction field of $R/(p)$, we have all coefficients of $f_D(t) = \sum_{i=1}^n a_i x^i$ lies in (p) but $a_n = 1$. Hence they actually lie in $\operatorname{Nil}(R)$.

Proposition 6. *There is a formal scheme $\operatorname{Div}_n^+(C)$ over X, which classifies all effective divisors of degree n on Y over X. Moreover, given a coordinate on C, $\operatorname{Div}_n^+(C) \cong X \times \widehat{\mathbb{A}}^n$.*

Suppose $n = 1$, a divisor D here is just a section of C over X. Conversely, a section is of course a closed subscheme of C which is finite and flat over X. Hence we have

$$\operatorname{Div}_1^+(C) = C$$

over X. The universal divisor D^u over $\operatorname{Div}_1^+(C) = C$ is a closed subscheme of $C \times_X C$. Given a coordinate X on C, the polynomial $f_{D^u}(t) = t - x$.

If we have two divisors D and D' over X, we define $D + D'$ to be the divisor corresponds to $f_D(t) f_{D'}(t)$. This defines a map

$$\operatorname{Div}_m^+(C) \times_X \operatorname{Div}_n^+(C) \to \operatorname{Div}_{m+n}^+(C),$$

which corresponds to

$$\begin{aligned} \mathcal{O}_X[\![x_0, x_1]\!] &\to \mathcal{O}_X[\![x, y]\!] \\ x_0 &\mapsto xy \\ x_1 &\mapsto x + y \end{aligned}$$

when m, n equals 1.

Hence we have a map $C_X^n/\Sigma_n \to \mathrm{Div}_n^+(C)$ for the commutativity of addition of divisors, which is an isomorphism.

Proposition 7. *The map $C_X^n/\Sigma_n \to \mathrm{Div}_n^+(C)$ is an isomorphism, and the universal divisor of degree n has the polynimial*

$$f_D(t) = \prod_k (x - a_k).$$

Proof. We work in the opposite category. $C_X^n/\Sigma_n \cong \mathrm{Spf}(\mathcal{O}_X[\![\sigma_1, \cdots, \sigma_n]\!])$ where σ_i is the $i'th$ elementary symmetric polynomial. The map $C_X^n \to \mathrm{Div}_n^+(C)$ induces

$$\mathcal{O}_{\mathrm{Div}_n^+(C)} = \mathcal{O}_X[\![a_1, \cdots, a_n]\!] \to \mathcal{O}_X[\![x_1, \cdots, x_n]\!]$$

with a_i sending to σ_i, the $i'th$ elementary polynomial. Therefore $\mathrm{Div}_n^+(C)$ is equal to C_X^n/Σ_n. □

Definition 10. *A subgroup of G is a divisor of G which is also a subgroup.*

Proposition 8. *Suppose K is a subgroup of G, then degree K is a power of p.*

Proof. Since K is a subgroup, we know that $\mathcal{O}_K \cong \mathrm{Spf}(\mathcal{O}_X[\![x]\!]/f_K(x))$. We also have

$$\begin{array}{ccc} K \times_X K & \longrightarrow & G \times_X G \\ \downarrow & & \downarrow \mu \\ K & \longrightarrow & G. \end{array}$$

The multiplication of K must factor through K. In another word,

$$\begin{array}{ccc} \mathcal{O}_X[\![t]\!] & \longrightarrow & \mathcal{O}_X[\![t]\!]/f_K(t) \\ \downarrow \mu^* & & \downarrow \\ \mathcal{O}_X[\![x,y]\!] & \longrightarrow & \mathcal{O}_X[\![x,y]\!]/(f_K(x), f_K(y)). \end{array}$$

This means

$$f_K(g(x,y)) = 0 \bmod f_K(x), f_K(y).$$

Now checking the coefficients like we already done after Example 3, the degree of f_K must be a power of p. □

Remark 6. *The group structure also requires that the identity lies in K, this is the same as*

$$X \to G$$

factors through

$$X \dashrightarrow K \longrightarrow G.$$

This is equivalent to require that $f_K(x) \in (x)$. Hence if K is a subgroup of G, we must have f_K is a polynomial of degree p^r and divided by x, i.e. it has no constant terms.

Proposition 9. *Suppose K is a subgroup of G with degree p^m, then $[p]_K^m = 0$. Hence $K \le G(m) = \ker([p]_G^m)$.*

Proof. For any R, $K(R)$ is a subgroup of $G(R) = \mathrm{Nil}(R)$, with elements satisfying the relation $f_K = 0$. All solutions of f_K in R are automatically nilpotent for all coefficients of f_K are nilpotent. Suppose there are some solutions, say α, not in R. We can embed $\mathrm{Nil}(R)$ into $\mathrm{Nil}(R[\alpha])$. So we can assume f_K has all solutions in R. Hence $K(R)$ is a group of order p^m. Thus any elements has order p^m. In another words

$$G(R) \xrightarrow{[p]^m} G(R),\ x \mapsto [p]^m(x) = x +_G \cdots +_G x$$

restricts to

$$K(R) \xrightarrow{[p]^m} K(R),\ x \mapsto [p]^m(x) = 0.$$

□

Having define subgroups of a formal group, we then consider the quotient groups. As in group theory, we define G/K to be the coequalizer

$$G \times_X K \overset{\mu}{\underset{\pi}{\rightrightarrows}} G \longrightarrow G/K.$$

On the level of functions, we have

$$\mathcal{O}_{G/K} \longrightarrow \mathcal{O}_G \overset{\mu^*}{\underset{\pi^*}{\rightrightarrows}} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_K.$$

Let x be a coordinate on G, and y be $N_\pi \mu^* x \in \mathcal{O}_G$.

Theorem. *Let K be a subgroup of G with degree p^m. The element y actually lies in $\mathcal{O}_{G/K}$, which satisfies*

(1) $y = x^{p^m}$ *mod* $\mathfrak{m}_X$.
(2) $\mathcal{O}_{G/K} = \mathcal{O}_X[\![y]\!]$
(3) G/K *has a natural structure as a formal group.*

Proof. First notice that there is an automorphism θ from $G \times_X K$ to it self, which sending (a, b) to $(a - b, b)$. Therefore $\pi = \mu\theta$, and π finite flat implies μ finite flat. Now we have two pullback diagrams with π' forget the third component.

$$\begin{CD} G \times_X K \times_X K @>{\mu \times 1}>> G \times_X K \\ @V{\pi'}VV @VV{\pi}V \\ G \times_X K @>{\mu}>> G \end{CD} \qquad \begin{CD} G \times_X K \times_X K @>{1 \times \mu}>> G \times_X K \\ @V{\pi'}VV @VV{\pi}V \\ G \times_X K @>{\pi}>> G \end{CD}$$

The second one is a pullback for there is a unique automorphism extending the following diagram

$$\begin{array}{ccc} G \times_X K \times_X K & & \\ \downarrow & \overset{1 \times \mu}{\searrow} & \\ G \times_X K \times_X K & \xrightarrow{\pi \times 1} & G \times_X K \end{array}$$

which sends (a, b, c) to $(a, b, b + c)$.

The maps involved in above diagrams are all finite flat maps, therefore we have

$$\mu^* N_\pi = N_{\pi'}(\mu \times 1)^* \quad , \quad \pi^* N_\pi = N_{\pi'}(1 \times \mu)^*.$$

Follow this and $\mu(1 \times \mu) = \mu(\mu \times 1)$ yields $y \in \mathcal{O}_{G/K}$.

For (a), let $j : K \to G$ be the inclusion, and $i : K \to G \times_X K$ with $i(a) = (0, a)$. Hence $j = \mu i$ and $\pi i = 0$. We have the following.

$$\begin{array}{ccccc} K & \xrightarrow{i} & G \times_X K & \xrightarrow{\mu} & G \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{0} & G & & \end{array}$$

Now $j^*y = i^*\mu^*y = i^*\pi^*y = 0^*y = 0$. This implies that y is divisible by f_K.

Recall that y is the norm of μ^*x under π. After mod $\mathfrak{m}_X$, $f_K(z)$ becomes z^{p^m} and we can write μ^*x as

$$x + a_1(x)z + a_2(x)z^2 + \cdots + a_{p^m-1}(x)z^{p^m-1} \in k[\![x]\!][z]/z^{p^m}$$

where k is the residue field of $\mathcal{O}_X$.

A direct computation implies

$$\mu^*x(1, z, \cdots, z^{p^m-1}) = (1, z, \cdots, z^{p^m-1}) \begin{bmatrix} x & & & \\ a_1(x) & x & & \\ \vdots & \vdots & \ddots & \\ a_{p^m-1}(x) & a_{p^m-2}(x) & \cdots & x \end{bmatrix}.$$

Therefore $y = N_\pi \mu^* x = x^{p^m}$ mod $\mathfrak{m}_X$. This completes (a). Moreover y is a unit multiple of $f_K(x)$.

For part (b), suppose $u \in \mathcal{O}_{G/K}$. Consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{G/K} & \longrightarrow & \mathcal{O}_G \\ {\scriptstyle 0^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_K \end{array}$$

The vertical map 0^* is just taking constant terms. From this we know $j^*(u - u(0)) = 0$. Hence $u - u(0)$ is divided by y. We can write $u = u(0) + u'y$ with $u' \in \mathcal{O}_G$. Since $u'y \in \mathcal{O}_{G/K}$, we conclude that $\pi^*(u'y) = \pi^*(u')\pi^*(y) = \mu^*(u'y) = \mu^*(u')\mu^*(y)$. The element $\pi^*(y) = \mu^*(y)$ is not a zero divisor, which implies $u' \in \mathcal{O}_{G/K}$. By induction, we have

$$\mathcal{O}_{G/K} = \mathcal{O}[\![y]\!].$$

Part (c) is obvious. G/K has the induced formal group law from G, with the coordinate y induced from x. And the projection $G \to G/K$ is the category cokernel of $K \to G$. $\square$