

# Barr-Beck Theorem, Morita theory and Brauer groups in $\infty$ -categories

Jiacheng Liang

Southern University of Science and Technology

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# Monadicity

In classical monad theory, given a monad  $T \in \text{Fun}(\mathcal{C}, \mathcal{C})$  it will induce a natural adjunction  $\mathcal{C} \rightleftarrows \text{LMod}_T(\mathcal{C})$ . The  $\text{LMod}_T(\mathcal{C})$  here is often denoted by  $\text{Alg}_T(\mathcal{C})$  in classical references.

## Definition

Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor. We will say  $G$  is monadic iff there exists a monad  $T \in \text{Fun}(\mathcal{C}, \mathcal{C})$  and an equivalence  $G_0 : \mathcal{D} \rightarrow \text{LMod}_T(\mathcal{C})$  such that  $G$  is equivalent to the composition of  $G_0$  with the forgetful functor  $\text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C}$ .

By remark above, any monadic functor is a right adjoint functor.

## Question

**Given an adjunction  $\mathcal{C} \rightleftarpoons_G^F \mathcal{D}$ , when is  $G$  monadic?**

We will see that the Barr-Beck Theorem provides a full answer.

# Barr-Beck Theorem

## Theorem (Classical Barr-Beck Theorem)

Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor which admits a left adjoint. Then the following are equivalent:

- 1 The functor  $G$  exhibits  $\mathcal{D}$  as monadic over  $\mathcal{C}$ .
- 2 There exists a monoidal category  $\mathcal{E}^\otimes$ , a left action of  $\mathcal{E}^\otimes$  on  $\mathcal{C}$ , an algebra object  $A \in \text{Alg}(\mathcal{E})$  and an equivalence  $G' : \mathcal{D} \simeq \text{LMod}_A(\mathcal{C})$  such that  $G$  is equivalent to the composition of  $G'$  with the forgetful functor  $\text{LMod}_A(\mathcal{C}) \rightarrow \mathcal{C}$ .
- 3 The functor  $G$  satisfies the following 2 conditions:
  - (a) The functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is conservative; that is, a morphism  $f : Y \rightarrow Y_0$  in  $\mathcal{D}$  is an equivalence if and only if  $G(f)$  is an equivalence in  $\mathcal{C}$ .
  - (b) Let  $V_1 \rightrightarrows V_0$  be a pair of morphisms of  $\mathcal{D}$  which is  $G$ -split. Then it admits a colimit in  $\mathcal{D}$ , and that colimit is preserved by  $G$ .

# $\infty$ -Barr-Beck Theorem

## Theorem ( $\infty$ -Barr-Beck Theorem)

Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor of  $\infty$ -categories which admits a left adjoint. Then the following are equivalent:

- 1 The functor  $G$  exhibits  $\mathcal{D}$  as monadic over  $\mathcal{C}$ .
- 2 There exists a monoidal  $\infty$ -category  $\mathcal{E}^\otimes$ , a left action of  $\mathcal{E}^\otimes$  on  $\mathcal{C}$ , an algebra object  $A \in \text{Alg}(\mathcal{E})$  and an equivalence  $G' : \mathcal{D} \simeq \text{LMod}_A(\mathcal{C})$  such that  $G$  is equivalent to the composition of  $G'$  with the forgetful functor  $\text{LMod}_A(\mathcal{C}) \rightarrow \mathcal{C}$ .
- 3 The functor  $G$  satisfies the following 2 conditions:
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  - (b) Let  $V_*$  be a simplicial object of  $\mathcal{D}$  which is  $G$ -split. Then it admits a colimit in  $\mathcal{D}$ , and that colimit is preserved by  $G$ .

# Examples of monadicity

## Example

1. Let  $\mathcal{C}^{\otimes}$  be a monoidal  $\infty$ -category. Then the forgetful functor  ${}_A \mathbf{BMod}_B(\mathcal{C}) \rightarrow \mathcal{C}$  is monadic, where the monad  $T$  is given by  $A \otimes (-) \otimes B$ .
2. Let  $\mathcal{C}, \mathcal{D}$  be presentable  $\infty$ -categories and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor which admits a left adjoint. If  $G$  is monadic, then so is  $Sp(G) : Sp(\mathcal{D}) \rightarrow Sp(\mathcal{C})$ .
3. By the fact that  $\Omega^\infty : Sp_{\geq 0} \rightarrow \mathcal{S}$  preserves sifted colimits (which includes geometric realization), the forgetful functor  $\mathbf{Alg}_{\mathbb{E}_k}(Sp_{\geq 0}) \rightarrow Sp_{\geq 0} \rightarrow \mathcal{S}$  is monadic.

In other words, we can identify connective  $\mathbb{E}_k$ -rings as spaces equipped with some additional structures, i.e.  $\mathbb{E}_k$ -(semi)ring space structure. Roughly speaking, it consists of an addition and multiplication which satisfy the axioms for a ring (commutative if  $k \geq 2$ ), up to coherent homotopy.

## Question

Morita theory began with a natural question: **To what extent does the module category  $\text{Mod}_R$  determine the ring  $R$  itself?** (Also known as recognition principles)

Firstly, we can consider the realization problem: when can a category  $\mathcal{C}$  be realized as some module category  $\text{Mod}_R$ ? That was answered by Schwede–Shikey.

Theorem (Schwede–Shikey 2003)

*Let  $\mathcal{C}$  be a stable  $\infty$ -category. Then  $\mathcal{C}$  is equivalent to  $\text{RMod}_R$ , for some  $\mathbb{E}_1$ -ring  $R$ , if and only if  $\mathcal{C}$  is presentable and there exists a compact object  $C \in \mathcal{C}$  which generates  $\mathcal{C}$  in the following sense: if  $D \in \mathcal{C}$  is an object having the property that  $\text{Ext}_{\mathcal{C}}^n(C, D) \simeq 0$  for all  $n \in \mathbb{Z}$ , then  $D \simeq 0$ .*

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## Theorem (Schwede–Shipley 2003)

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# Higher Morita theory

Secondly, we wish characterize functors between module categories. We begin with a classical Morita theorem.

## Theorem (classical Morita theorem)

Let  $R$  and  $R'$  be associative rings, and let  $\mathbf{LFun}(R\mathbf{Mod}_R, R\mathbf{Mod}_{R'})$  be the category of functors from  $R\mathbf{Mod}_R$  to  $R\mathbf{Mod}_{R'}$  which preserve small colimits. Then the relative tensor product functor  $\otimes_R : R\mathbf{Mod}_R \times_R R\mathbf{Mod}_{R'} \rightarrow R\mathbf{Mod}_{R'}$  induces an equivalence of categories

$$R\mathbf{Mod}_{R'} \rightarrow \mathbf{LFun}(R\mathbf{Mod}_R, R\mathbf{Mod}_{R'}).$$

By the theorem above we can see two equivalent module categories does not imply two equivalent rings.

Actually, this leads us to the definition of Morita equivalence between rings.



# Morita equivalence

## Definition (Morita equivalence)

Let  $\mathcal{C}^\otimes$  be a monoidal  $\infty$ -category compatible with geometric realization. Given  $R, R' \in \text{Alg}(\mathcal{C})$ , we say that they are Morita equivalent iff there exists  ${}_R M_{R'} \in {}_R \text{BMod}_{R'}$  and  ${}_{R'} N_R \in {}_{R'} \text{BMod}_R$  such that  ${}_R M \otimes_{R'} N_R \simeq {}_R R_R$  in  ${}_R \text{BMod}_R$  and that  ${}_{R'} N \otimes_R M_{R'} \simeq {}_{R'} R'_{R'}$  in  ${}_{R'} \text{BMod}_{R'}$ .

## Definition (Morita category)

Let  $\mathcal{K}$  be a small collection of simplicial sets which includes  $\mathbf{N}(\Delta)^{op}$  and  $\mathcal{C}^\otimes \rightarrow \text{Ass}^\otimes$  be a monoidal  $\infty$ -category compatible with  $\mathcal{K}$ -colimits. For every algebra object  $A \in \text{Alg}(\mathcal{C})$ , the  $\infty$ -category  $\text{RMod}_A(\mathcal{C})$  is left-tensored over  $\mathcal{C}$ , and can therefore be identified with a left  $\mathcal{C}$ -module object of  $\text{Cat}_\infty(\mathcal{K})$ .

We let  $\text{Morita}(\mathcal{C})$  denote the full subcategory of  $\text{LMod}_{\mathcal{C}}(\text{Cat}_\infty(\mathcal{K}))$  spanned by objects of the form  $\text{RMod}_A(\mathcal{C})$ , where  $A \in \text{Alg}(\mathcal{C})$ . We will refer to  $\text{Morita}(\mathcal{C})$  as the Morita  $\infty$ -category of  $\mathcal{C}$ .

## Definition

1. Let  $\text{Cat}_\infty^{\text{Alg}}(\mathcal{K})$  be the large  $\infty$ -category (informally) described as follows:
  - objects are pairs  $(\mathcal{C}^\otimes, A)$  where  $\mathcal{C}^\otimes$  is a monoidal  $\infty$ -category compatible with  $\mathcal{K}$ -colimits and  $A \in \text{Alg}(\mathcal{C})$ .
  - a morphism from  $(\mathcal{C}^\otimes, A)$  to  $(\mathcal{D}^\otimes, B)$  is a monoidal functor  $F : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$  preserving  $\mathcal{K}$ -colimits and a morphism  $F(A) \rightarrow B$  in  $\text{Alg}(\mathcal{D})$ .
2. Let  $\text{Cat}_\infty^{\text{Mod}}(\mathcal{K})$  be the large  $\infty$ -category (informally) described as follows:
  - objects are pairs  $(\mathcal{C}^\otimes, \mathcal{M})$  where  $\mathcal{M}$  is left-tensored over  $\mathcal{C}$  where tensor product is compatible with  $\mathcal{K}$ -colimits.
  - a morphism from  $(\mathcal{C}^\otimes, \mathcal{M})$  to  $(\mathcal{D}^\otimes, \mathcal{N})$  is a monoidal functor  $F : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$  preserving  $\mathcal{K}$ -colimits and a  $\mathcal{C}$ -linear functor  $\mathcal{M} \rightarrow \mathcal{N}$  in  $\text{Alg}(\mathcal{D})$ .

## Proposition

The natural forgetful functors  $\text{Cat}_\infty^{\text{Alg}}(\mathcal{K}) \rightarrow \text{Mon}_{\text{Ass}}^{\mathcal{K}}(\text{Cat}_\infty)$  and  $\text{Cat}_\infty^{\text{Mod}}(\mathcal{K}) \rightarrow \text{Mon}_{\text{Ass}}^{\mathcal{K}}(\text{Cat}_\infty)$  are coCartesian fibrations.

We can define a functor  $\Theta : \text{Cat}_{\infty}^{\text{Alg}}(\mathcal{K}) \rightarrow \text{Cat}_{\infty}^{\text{Mod}}(\mathcal{K})$  by  $(\mathcal{C}^{\otimes}, A) \mapsto \text{RMod}_A(\mathcal{C})$ , where the  $\infty$ -category  $\text{RMod}_A(\mathcal{C})$  of right  $A$ -module objects of  $\mathcal{C}$  is viewed as an  $\infty$ -category left-tensored over  $\mathcal{C}$ .

### Proposition

The construction  $\Theta$  above is a coCartesian-preserving functor,

$$\begin{array}{ccc}
 \text{Cat}_{\infty}^{\text{Alg}}(\mathcal{K}) & \xrightarrow{\quad \Theta \quad} & \text{Cat}_{\infty}^{\text{Mod}}(\mathcal{K}) \\
 & \searrow & \swarrow \\
 & \text{Mon}_{\text{Ass}}^{\mathcal{K}}(\text{Cat}_{\infty}) &
 \end{array}$$

whose restriction on the fiber of any  $\mathcal{C}^{\otimes} \in \text{Mon}_{\text{Ass}}^{\mathcal{K}}(\text{Cat}_{\infty})$  gives a functor  $\text{Alg}(\mathcal{C}) \xrightarrow{\Theta_{\mathcal{C}}} \text{LMod}_{\mathcal{C}}(\text{Cat}_{\infty}^{\mathcal{K}})$ .

The essential image of  $\Theta_{\mathcal{C}}$  is exactly the Morita category of  $\mathcal{C}$ . Roughly speaking, Morita theory is to study the properties of the functor  $\Theta$ .

## Lemma

Let  $\mathcal{K}$  be a small collection of simplicial sets. Let  $\mathcal{S}(\mathcal{K}) \subset \mathcal{S}$  be the maximal full subcategory containing  $\Delta^0$  and closed under  $\mathcal{K}$ -colimits, which inherits a Cartesian monoidal structure from  $\mathcal{S}$  and we denote as  $\mathcal{S}(\mathcal{K})^\times$ . Then the pair  $(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})$  is an initial object of  $\text{Cat}_\infty^{\text{Alg}}(\mathcal{K})$ .

## Definition

It follows the Lemma that the forgetful functor  $\theta : \text{Cat}_\infty^{\text{Alg}}(\mathcal{K})_{(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})/} \rightarrow \text{Cat}_\infty^{\text{Alg}}(\mathcal{K})$  is a trivial Kan fibration. We let  $\Theta_*$  denote the composition

$$\text{Cat}_\infty^{\text{Alg}}(\mathcal{K}) \simeq \text{Cat}_\infty^{\text{Alg}}(\mathcal{K})_{(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})/} \xrightarrow{\Theta} \text{Cat}_\infty^{\text{Mod}}(\mathcal{K})_{\mathfrak{M}/}$$

where the first map is given by a section of  $\theta$  and  $\mathfrak{M} = \Theta(\mathcal{S}(\mathcal{K})^\times, \mathbf{1}) = (\mathcal{S}(\mathcal{K})^\times, \mathcal{S}(\mathcal{K}))$ .

An object of the  $\infty$ -category  $\text{Cat}_\infty^{\text{Mod}}(\mathcal{K})_{\mathfrak{M}/}$  is given by a morphism  $(\mathcal{S}(\mathcal{K})^\times, \mathcal{S}(\mathcal{K})) \rightarrow (\mathcal{C}^\otimes, \mathcal{M})$  in  $\text{Cat}_\infty^{\text{Mod}}(\mathcal{K})$ , given by the unique monoidal functor  $\mathcal{S}(\mathcal{K})^\times \rightarrow \mathcal{C}^\otimes$  which preserves  $\mathcal{K}$ -indexed colimits together with a functor  $f : \mathcal{S}(\mathcal{K}) \rightarrow \mathcal{M}$  which preserves  $\mathcal{K}$ -indexed colimits. Such a functor is determined uniquely up to equivalence by the object  $f(\Delta^0) \in \mathcal{M}$ .

Consequently, we can informally regard  $\text{Cat}_\infty^{\text{Mod}}(\mathcal{K})_{\mathfrak{M}/}$  as an  $\infty$ -category whose objects are triples  $(\mathcal{C}^\otimes, \mathcal{M}, M)$ , where  $(\mathcal{C}^\otimes, \mathcal{M}) \in \text{Cat}_\infty^{\text{Mod}}(\mathcal{K})$  and  $M \in \mathcal{M}$  is an object.

### Theorem

Let  $\mathcal{K}$  be a small collection of simplicial sets containing  $\mathbf{N}(\Delta)^{\text{op}}$ . Then the functor  $\Theta_* : \text{Cat}_\infty^{\text{Alg}}(\mathcal{K}) \rightarrow \text{Cat}_\infty^{\text{Mod}}(\mathcal{K})_{\mathfrak{M}/}$  is fully faithful. Therefore when it restricts on any fiber  $\mathcal{C}^\otimes \in \text{Mon}_{\text{Ass}}^{\mathcal{K}}(\text{Cat}_\infty)$ , the

$$\text{Alg}(\mathcal{C}) \rightarrow \text{LMod}_{\mathcal{C}}(\text{Cat}_\infty^{\mathcal{K}})_{\mathcal{C}/}$$

informally given by  $A \mapsto \mathcal{C} \xrightarrow{-\otimes A} \text{RMod}_A(\mathcal{C})$ , is also fully faithful.

In this sense of the full subcategory, algebras are determined by their module categories.

# Essential image of $\Theta$

By Barr-Beck theorem it is not hard to describe the essential image of  $\Theta$ .

## Theorem

Let  $\mathcal{C}$  be a monoidal  $\infty$ -category. Assume that  $\mathcal{C}$  admits  $\mathbf{N}(\Delta)^{op}$ -colimits and that the tensor product  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  preserves  $\mathbf{N}(\Delta)^{op}$ -colimits. Let  $\mathcal{M}$  be an  $\infty$ -category left-tensored over  $\mathcal{C}$  and let  $M \in \mathcal{M}$  be an object. Then there exists an algebra object  $A \in \text{Alg}(\mathcal{C})$  and an equivalence  $\text{RMod}_A(\mathcal{C}) \simeq \mathcal{M}$  of  $\infty$ -categories left-tensored over  $\mathcal{C}$  which carries  $A$  to  $M$  if and only if the following conditions are satisfied:

- (1) The  $\infty$ -category  $\mathcal{M}$  admits  $\mathbf{N}(\Delta)^{op}$ -colimits. And the action map  $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$  preserves  $\mathbf{N}(\Delta)^{op}$ -colimits.
- (2) The functor  $F : \mathcal{C} \rightarrow \mathcal{M}$  given by  $F(C) = C \otimes M$  admits a right adjoint  $G$  such that  $G$  is conservative and preserves  $\mathbf{N}(\Delta)^{op}$ -colimits.
- (3) For every object  $N \in \mathcal{M}$  and every object  $C \in \mathcal{C}$ , the evident map  $F(C \otimes G(N)) \simeq C \otimes G(N) \otimes M \simeq C \otimes FG(N) \rightarrow C \otimes N$  is adjoint to an equivalence  $C \otimes G(N) \xrightarrow{\sim} G(C \otimes N)$ .

In this case, we actually have  $A \simeq \text{End}_{\mathcal{M}}(M)$ .

# Universal properties of $R\text{Mod}_A(\mathcal{C})$

## Theorem

Let  $\mathcal{K}$  be a collection of simplicial sets which includes  $\mathbf{N}(\Delta)^{op}$ , let  $\mathcal{C}^{\otimes}$  be a monoidal  $\infty$ -category, and  $\mathcal{M}$  an  $\infty$ -category left-tensored over  $\mathcal{C}$ . Assume that  $\mathcal{C}$  and  $\mathcal{M}$  admit  $\mathcal{K}$ -indexed colimits, and that the tensor product functors  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and  $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$  preserve  $\mathcal{K}$ -indexed colimits separately in each variable. Let  $A$  be an algebra object of  $\mathcal{C}$ , and let  $\theta$  denote the composition

$$\begin{aligned} \text{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\text{RMod}_A(\mathcal{C}), \mathcal{M}) &\subseteq \text{LinFun}_{\mathcal{C}}(\text{RMod}_A(\mathcal{C}), \mathcal{M}) \\ &\xrightarrow{\theta'} \text{Fun}(\text{LMod}_A(\text{RMod}_A(\mathcal{C})), \text{LMod}_A(\mathcal{M})) \\ &\xrightarrow{\theta''} \text{LMod}_A(\mathcal{M}), \end{aligned}$$

where  $\theta''$  is given by evaluation at the  $A$ -bimodule given by  ${}_A A_A$ . Then  $\theta$  is an equivalence of  $\infty$ -categories.

# Universal properties of $R\text{Mod}_A(\mathcal{C})$

## Corollary

Particularly, when  $\mathcal{M} = R\text{Mod}_B(\mathcal{C})$  we have an equivalence of  $\infty$ -categories  $\text{LinFunc}_{\mathcal{C}}(R\text{Mod}_A(\mathcal{C}), R\text{Mod}_B(\mathcal{C})) \simeq \text{LMod}_A(R\text{Mod}_B(\mathcal{C})) \simeq {}_A\text{BMod}_B(\mathcal{C})$ .

That is, every  $\mathcal{C}$ -linear functor from  $R\text{Mod}_A(\mathcal{C})$  to  $R\text{Mod}_B(\mathcal{C})$  which preserves  $\mathcal{K}$ -indexed colimits is given by the formula  $M \mapsto M \otimes_A K$ , for some bimodule object  ${}_A K_B \in {}_A\text{BMod}_B(\mathcal{C})$ . It follows from this description that  $\text{Morita}(\mathcal{C})$  is independent of the choice of  $\mathcal{K}$ , so long as  $\mathcal{K}$  includes  $\mathbf{N}(\Delta)^{op}$ .

## Proposition

*The equivalence above satisfies the composition law*

$$\begin{array}{ccc} {}_C\text{BMod}_B(\mathcal{C}) \times {}_B\text{BMod}_A(\mathcal{C}) & \longrightarrow & \text{LinFunc}_{\mathcal{C}}(R\text{Mod}_C(\mathcal{C}), R\text{Mod}_B(\mathcal{C})) \times \text{LinFunc}_{\mathcal{C}}(R\text{Mod}_B(\mathcal{C}), R\text{Mod}_A(\mathcal{C})) \\ \downarrow & & \downarrow \\ {}_C\text{BMod}_A(\mathcal{C}) & \longrightarrow & \text{LinFunc}_{\mathcal{C}}(R\text{Mod}_C(\mathcal{C}), R\text{Mod}_A(\mathcal{C})) \end{array}$$



Now we can give a good description of Morita category.

### Definition

Define a new  $h\mathcal{S}$ -enriched category  $\text{Morita}'(\mathcal{C})$  as follows:

- objects are algebra objects  $A \in \text{Alg}(\mathcal{C})$ .
- Given a pair of objects  $A, B \in \text{Alg}(\mathcal{C})$ , the mapping space  $\text{Map}_{\text{Morita}'(\mathcal{C})}(A, B)$  can be identified with the Kan complex  ${}_A \text{BMod}_B(\mathcal{C})^{\simeq}$ .
- Given a triple of objects  $A, B, C \in \text{Alg}(\mathcal{C})$ , the composition law

$${}_C \text{BMod}_B(\mathcal{C})^{\simeq} \times {}_B \text{BMod}_A(\mathcal{C})^{\simeq} \rightarrow {}_C \text{BMod}_A(\mathcal{C})^{\simeq}$$

is given by  $(M, N) \mapsto M \otimes_B N$ .

### Corollary

The natural enriched functor  $\text{Morita}'(\mathcal{C}) \rightarrow \underline{h\text{Morita}}(\mathcal{C})$  is an equivalence of  $h\mathcal{S}$ -enriched categories.

# Brauer $\infty$ -group

Actually,  $Morita(\mathcal{C})$  is the underlying  $(\infty, 1)$ -category of the  $(\infty, 2)$ -category  $MORITA(\mathcal{C})$ , whose mapping  $\infty$ -categories are  ${}_A BMod_B(\mathcal{C})$ .

## Definition (Brauer space)

We define the (big) Brauer space  $\mathbf{Br}(\mathcal{C})$  with respect to  $\mathcal{C}^{\otimes}$  as  $Morita(\mathcal{C})^{\simeq}$ , i.e. the underlying  $(\infty, 0)$ -category, also known as the maximal Kan complex.

It is often known that there is a group structure, called the Brauer group. Actually we will see that  $\mathbf{Br}(\mathcal{C})$  has a natural group-like  $\mathbb{E}_{\infty}$ -structure, therefore  $\pi_0 \mathbf{Br}(\mathcal{C})$  is indeed a (big) group. But before that, we need to introduce monoidal Morita theory.

## Proposition

The  $\Theta : \text{Cat}_{\infty}^{\text{Alg}}(\mathcal{K}) \rightarrow \text{Cat}_{\infty}^{\text{Mod}}(\mathcal{K})$  can be naturally enhanced to a symmetric monoidal functor  $\Theta^{\otimes} : \text{Cat}_{\infty}^{\text{Alg}}(\mathcal{K})^{\otimes} \rightarrow \text{Cat}_{\infty}^{\text{Mod}}(\mathcal{K})^{\otimes}$ , whose restriction on each fiber of  $\mathcal{C}^{\otimes}$  is a symmetric monoidal functor  $\Theta_{\mathcal{C}}^{\otimes} : \text{Alg}(\mathcal{C})^{\otimes} \rightarrow \text{LMod}_{\mathcal{C}}(\text{Cat}_{\infty}(\mathcal{K}))^{\otimes}$ , informally given by  $A \mapsto RMod_A(\mathcal{C})^{\otimes}$ .

## Definition

The monoidal enhancement makes  $\mathbf{Br}(\mathcal{C}) \subset \mathit{Morita}(\mathcal{C}) \subset \mathit{LMod}_{\mathcal{C}}(\mathit{Cat}_{\infty}(\mathcal{K}))$  inherit a natural symmetric monoidal structure, denoted as  $\mathbf{Br}(\mathcal{C})^{\otimes}$  and  $\mathit{Morita}(\mathcal{C})^{\otimes}$ .

By the natural equivalence  $\mathit{CAlg}(\mathcal{S}) \simeq \mathit{Mon}(\mathcal{S})$  between  $\mathbb{E}_{\infty}$ -spaces and symmetric monoidal groupoids, the  $\mathbf{Br}(\mathcal{C})^{\otimes}$  naturally corresponds an  $\mathbb{E}_{\infty}$ -space.

Now we turn to the case of  $\mathbb{E}_{\infty}$ -rings.

## Definition

Let  $R$  be an  $\mathbb{E}_{\infty}$ -ring. Taking  $\mathcal{C}^{\otimes} = \mathit{Mod}_R^{\otimes}$  we get a symmetric monoidal functor  $\Theta_{\mathit{Mod}_R}^{\otimes} : \mathit{Alg}_R^{\otimes} = \mathit{Alg}(\mathit{Mod}_R)^{\otimes} \rightarrow \mathit{LMod}_{\mathit{Mod}_R}(\mathit{Pr}^L)^{\otimes} = \mathit{LMod}_{\mathit{Mod}_R}(\mathit{Pr}_{st}^L)^{\otimes} = \mathit{Cat}_R^{\otimes}$ , where the latter is often called  $R$ -linear categories, informally given by  $A \mapsto \mathit{RMod}_A^{\otimes}$ .

Because all  $\mathit{RMod}_A$  are compactly generated,  $\Theta_{\mathit{Mod}_R}^{\otimes}$  factors through  $\mathit{Alg}_R^{\otimes} \rightarrow \mathit{Cat}_{R,\omega}^{\otimes}$ , where  $\mathit{Cat}_{R,\omega}^{\otimes}$  is the category of compactly generated  $R$ -linear categories.

# Azumaya algebras

Now we introduce Azumaya algebras and (small) Brauer groups.

## Definition

Let  $R$  be an  $\mathbb{E}_\infty$ -ring. An  $R$ -algebra  $A$  is an Azumaya  $R$ -algebra if  $A$  is a compact generator of  $\text{Mod}_R$  and if the natural  $R$ -algebra map

$$A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$$

is an equivalence of  $R$ -algebras.

Note that if  $A$  is an Azumaya  $R$ -algebra, then, by definition,  $A \otimes_R A^{\text{op}}$  is Morita equivalent to  $R$ . The standard example of an Azumaya algebra is the endomorphism algebra  $\text{End}_R(P)$  of a compact generator  $P \in \text{Mod}_R$ .

## Theorem (Toën 2012)

*If  $R = \mathbb{H}k$ , where  $k$  is an algebraically closed field, then every Azumaya  $R$ -algebra is Morita equivalent to  $R$ .*

# Azumaya algebras

## Proposition

Let  $A$  be an  $R$ -algebra. Then,  $A$  is compact in  $\text{Alg}_R$  if and only if  $\text{Mod}_A$  is compact in the  $\text{Cat}_{R,\omega}$ .

## Corollary

Compactness in  $\text{Alg}_R$  is a Morita-invariant property.

## Theorem (Antieau–Gepner 2012)

Let  $\mathcal{C} \in \text{Cat}_{R,\omega}$ . Then it is invertible in  $\text{Cat}_{R,\omega}$  if and only if  $\mathcal{C}$  is equivalent to  $\text{Mod}_A$  for an Azumaya  $R$ -algebra  $A$ .

This leads to small Brauer groups.

## Proposition

The image of Azumaya algebras under  $\Theta_{\text{Mod}_R}^{\otimes} : \text{Alg}_R^{\otimes} \rightarrow \text{Cat}_{R,\omega}^{\otimes}$  is exactly those invertible objects  $\text{Cat}_{R,\omega}$ .

## Definition

We define the (small) Brauer space  $Br(R)$  as the essential image of Azumaya algebras in  $\mathbf{Br}(R)$ . By the previous argument, we have the following chain of symmetric submonoidal categories.

$$Br(R)^{\otimes} \subset \mathbf{Br}(R)^{\otimes} \subset Morita(R)^{\otimes} \subset Cat_R^{\otimes}$$

That makes  $Br(R)$  become an  $\mathbb{E}_{\infty}$ -space. And we have  $Br(R) \simeq \text{Pic}(\text{Cat}_{R,\omega})$  as group-like  $\mathbb{E}_{\infty}$ -spaces.

## Theorem (Antieau–Gepner 2012)

*For  $R$  a connective  $E_{\infty}$  ring, any Azumaya  $R$ -algebra  $A$  is étale locally trivial: there is an étale cover  $R \rightarrow S$  such that  $A \otimes_R S$  is morita equivalent to  $S$ .*

## Theorem (Antieau–Gepner 2012)

*For  $R$  a connective  $E_{\infty}$  ring, the functor  $Br : \text{CAlg}_R^{\geq 0} \rightarrow \text{Gpd}_{\infty}$  restricting on connective  $\mathbb{E}_{\infty}$ - $R$ -algebras is a sheaf for the étale topology.*

## Theorem (Antieau–Gepner 2012)

Let  $X$  be an object of  $\mathrm{Shv}_R^{\mathrm{et}}$ . Then, there is a conditionally convergent spectral sequence

$$E_2^{p,q} = \begin{cases} H_{\mathrm{et}}^p(X, \pi_q Br) & p \leq q \\ 0 & p > q \end{cases} \Rightarrow \pi_{q-p} Br(X)$$

with differentials  $d_r$  of degree  $(r, r-1)$ . If  $X$  is affine or discrete, then the spectral sequence converges completely.

Proof. Because the Brauer sheaf  $Br$  is hypercomplete, the map from  $Br$  to the limit of its Postnikov tower  $Br \rightarrow \lim_n \tau_{\leq n} Br$  is an equivalence. Taking sections preserves limits, so that

$$Br(X) \rightarrow \lim_n (\tau_{\leq n} Br)(X)$$

is also an equivalence. Thus,  $Br(X)$  is the limit of a tower, and to any such tower there is an associated spectral sequence.

# Calculation

If  $X$  is affine or discrete, then the spectral sequence degenerates at some finite page. And if  $X$  is discrete the spectral sequence collapses entirely at the  $E_2$ -page. So, suppose that  $X = \text{Spec } S$ . Then,  $Br(X)$  can be computed on the small étale site on  $\text{Spec } S$ . But, as mentioned above, this site is the nerve of a discrete category, the small étale site on  $\text{Spec } \pi_0 S$ . Therefore,  $H_{\text{et}}^p(\text{Spec } S, \pi_q Br) \cong H_{\text{et}}^p(\text{Spec } \pi_0 S, \pi_q Br)$ .

## Corollary (Antieau–Gepner 2012)

If  $R$  is a connective  $\mathbb{E}_\infty$ -ring, then the homotopy groups of  $Br(R)$  are described by

$$\pi_k Br(R) \cong \begin{cases} H_{\text{et}}^1(\text{Spec } \pi_0 R, \mathbb{Z}) \times H_{\text{et}}^2(\text{Spec } \pi_0 R, \mathbb{G}_m) & k = 0 \\ H_{\text{et}}^0(\text{Spec } \pi_0 R, \mathbb{Z}) \times H_{\text{et}}^1(\text{Spec } \pi_0 R, \mathbb{G}_m) & k = 1 \\ \pi_0 R^\times & k = 2 \\ \pi_{k-2} R & k \geq 3. \end{cases}$$

## Corollary (Antieau–Gepner 2012)

The Brauer  $\mathbb{E}_\infty$ -group  $Br(\mathbb{S})$  of the sphere spectrum  $\mathbb{S}$  is zero, i.e. its underlying space is contractible.