An introduction to singularity theory, Milnor's fibration theorem and Brieskorn's construction of exotic spheres

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Singular point of a function

We only treat complex analysis function in this talk.

Definition

Given a function $f : \mathbb{C}^n \to \mathbb{C}$, the point $\mathbf{z}$ is called a **regular point** of f if its gradient $\nabla f = (\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n})\big|_{\mathbf{z}} \neq 0$. Otherwise, the point $\mathbf{z}$ is said to be a **critical point** of function f if all the derivatives are equal to zero. The value that the function takes at a critical point is called a critical value.

The regular points may be think as good points and critical points are bad points. Then we can continue to divide the critical points into not very bad (nondegenerate or generic) critical points and worse (degenerate) critical points.

Definition

A critical point is said to be **nondegnerate** (or a Morse critical point) if the Hessian matrix $\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right)_{i,j}$ at that point is a nondegenerate quadratic form.

Equivalence of critical points

We are only concerned with the local behavior around a critical point, denote $\mathcal{O}_n$ the set of holomorphic function germs f at the point $\mathbf{0} \in \mathbb{C}^n$, and $\mathcal{D}_n$ the group of germs of biholomorphic maps keep zero $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. $\mathcal{D}_n$ acts on $\mathcal{O}_n$ by $g \cdot f = f \circ g^{-1}$.

Definition

Two function germs at zero are said to be **equivalent** if one is taken into the other by a biholomorphic change of coordinates that keeps the point zero fixed.

Definition

Two critical points are said to be equivalent if the function-germs that define them are equivalent. The equivalence class of a function-germ at a critical point is called a **singularity**.

Morse Lemma

Example

All regular points are equivalent to the projection on one coordinate $f(z_1, z_2, \cdots) = z_1$. And $f_1 = x^m$ is equivalent to $f_2 = cx^m$.

The behavior of a function in the neighborhood of a nondegenerate critical point is described by the **Morse lemma**.

Lemma (Morse)

In a neighborhood of a nondegenerate critical point $a \in \mathbb{C}$ of the function f there exists a coordinate in which f has the form $f(\mathbf{z}) = z_1^2 + \cdots + z_n^2 + f(a)$.

For degenerate critical points, a generalization of the preceding result holds, the parametric Morse lemma.

Theorem (Arnol'd)

In a neighborhood of the critical point $\mathbf{0}$ *of corank* k *a holomorphic function* $f : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ *is equivalent to a function of the form*

$$f(\mathbf{z}) = \varphi(z_1, \cdots, z_k) + z_{k+1}^2 + \cdots + z_n^2,$$

where the second differential of φ *at zero is equal to zero, i.e.* φ *is at least* 3*-order infinitesimal.*

Stably equivalence

Previous theorem permits one to define an equivalence relation for critical points of functions of different numbers of variables.

Definition

Two function-germs $f : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}, 0)$ are said to be **stably equivalent** if they become equivalent after the addition of nondegenerate quadratic forms in supplementary variables:

$$f(z_1, \cdots, z_n) + z_{n+1}^2 + \cdots + z_l^2 \sim g(z_1, \cdots, z_m) + z_{m+1}^2 + \cdots + z_l^2.$$

Two functions of the same number of variables are stably equivalent if and only if they are equivalent.

Example

The function-germs $f(x) = x^3$ and $g(x, y, z) = x^3 + yz$ are stably equivalent at zero.

Invariants of singularity (Corank)

We want to find some invariants of a singular point under equivalent or stably equivalent before we can distinguish different singularities.

Definition

The **corank** of a critical point of a function is corank (dimension – rank) of the Hessian matrix at the critical point.

Example

The corank of any Morse critical point is equal to zero. The corank of the critical point 0 of the function $f = x_1^3 + x_2^2 + \cdots + x_n^2$ is equal to 1, since

$$H(f) = \begin{pmatrix} 6x_1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}_{\mathbf{x}=0} = \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}$$

Invariants of singularity (Multiplicity)

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with a critical point at zero. Let us consider the gradient ideal $I_{\nabla f} = \mathcal{O}_n \langle \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \rangle$ generated by the partial derivatives of f.

Definition

The **multiplicity** $\mu(f)$ (or Milnor number) of the critical point of the germ $f \in \mathcal{O}_n$ is the dimension of its local algebra regarded as a complex vector space $\dim_{\mathbb{C}}(\mathcal{O}_n / I_{\nabla f})$.

Theorem

A critical point is of ***finite multiplicity*** $\mu(f) < \infty$ *iff* ***isolated*** *(there is no other critical point in its neighborhood).*

Example

The function $f(z) = z^3$ has at zero an isolated critical point of multiplicity 2. The function $g(x, y) = xy^2$ has a nonisolated critical point at every point of the x-axis.

Invariants of singularity (Multiplicity) 2

Theorem

The ***multiplicity*** *of an* ***isolated*** *critical point is equal to the* ***number of Morse critical points*** *into which it decomposes under a small perturbation of the function.*

Theorem

If holomorphic function f has a an ***isolated*** *critical point at $\mathbf{z}_0$. The multiplicity $\mu(f)$ of $\mathbf{z}_0$ is the* ***mapping degree*** *of the map:*

$$\mathbf{z} \mapsto \frac{\nabla f(\mathbf{z})}{||\nabla f(\mathbf{z})||}$$

from a small sphere $S(\epsilon)$ centered at $\mathbf{z}_0$ to the unit sphere of $\mathbb{C}^n$.

k-jet

Definition

For a function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denote by $j^k_a(f)$ its k**-jet** at the point a. If one fixes a coordinate system on $\mathbb{C}^n$, then the k-jet can be thought of as the Taylor polynomial of degree k. A k-jet of a function is said to be **sufficient** at a if it determines the germ of that function at a up to equivalence.

Example

As follows from the Morse lemma, at a nondegenerate critical point a function is equivalent to its Taylor polynomial of degree two. Hence, the 2-jet of a function at a nondegenerate critical point is sufficient.

Theorem (Tougeron)

The $(\mu + 1)$-jet of a function at an isolated critical point of multiplicity μ is sufficient, i.e. this function is always equivalent to a polynomial.

Thus, the problem of classifying isolated critical points reduces to **algebraic problems** concerning the action of finite-dimensional Lie groups on finite-dimensional spaces of jets.

Definition

Let G be a Lie group acting on a manifold M, and f is a point of M. The **modality** of the point $f \in M$ under the action of the Lie group G is defined to be the least number m such that a sufficiently small neighborhood of f is covered by a finite number of m-parameter families of orbits.

Deformation of a critical point

Definition

A **deformation** with base $\Lambda = \mathbb{C}^l$ of the germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is the germ at zero of a smooth map $F : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ such that $F(x, 0) \equiv f(x)$. A deformation F' is equivalent to F if $F'(x, \lambda) = F(g(x, \lambda), \lambda)$, where $g : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to (\mathbb{C}^n, 0)$, with $g(x, 0) \equiv x$, is a smooth germ (a family of diffeomorphisms depending on the parameter $\lambda \in \Lambda$).

Definition

A deformation $F(x, \lambda)$ of the germ f(x) is said to be **versal** if every deformation F' of $f(x)$ can be represented in the form
$F'(x, \lambda') = F(g(x, \lambda'), \theta(\lambda')),\ g(x, 0) \equiv 0,\ \theta(0) = 0$, where $\theta : (\mathbb{C}^{l'}, 0) \to (\mathbb{C}^l, 0)$ is a smooth germ. i.e., if every deformation of $f(x)$ is equivalent to a deformation induced from F.

Invariants of singularity (Modality) 2

The group of germs of diffeomorphisms $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ acts on the function space $\mathfrak{m} = \mathcal{O}_n\langle z_1, \cdots, z_n\rangle \subset \mathcal{O}_n$ of germs of critical points, and consequently also on the k-jet space $j^k(\mathfrak{m})$. To avoid the difficulties inherent to the infinite-dimensional moduli space of holomorphic maps, let us define the modality of a germ f as the modality of its k-jet for sufficiently large k.

Definition

The **modality** m of a function-germ f is the modality of any of its jets $j^k(f)$, such that $k \geq \mu(f) + 1$. The functions of modality $m = 0, 1, 2$ are called respectively **simple**, **unimodal**, and **bimodal functions**.

The point is simple if a neighborhood of it is intersected by finitely many orbits.

Theorem

The three invariants of a singularity are related by $\mu = c + m + 1$.

Classification of low modality (simple)

A surprising conclusion of the computations carried out in the classification of singularities is that the most natural results are obtained not in classifying singularities by their corank c or multiplicity μ, but rather in classifying the singularities of low modality m.

Theorem (Arnol'd 72)

Up to stable equivalence, the simple germs with $m = 0$ are exhausted by the following list. There are two infinite series, A_k and D_k , and three exceptional singularities, E_6, E_7 and E_8:

$A_k,\ k \geq 1$	$D_k,\ k \geq 4$	E_6	E_7	E_8
x^{k+1}	$x^2y + y^{k-1}$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$

This is also called elliptic case.

Classification of low modality (unimodal)

Theorem

Up to stable equivalence, the unimodal germs with $m = 1$ are exhausted by the three-index series of one-parameter families, a three-index series of hyperbolic singularities, and 14 families of exceptional singularities.

Parabolic:

P_8	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$
X_9	$x^4 + y^4 + ax^2y^2$	$a^2 \neq 4$
J_{10}	$x^3 + y^6 + ax^2y^2$	$4a^3 + 27 \neq 0$

Hyperbolic:

$$T_{p,q,r}\colon x^p + y^q + z^r + axyz, \quad a \neq 0, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Classification of low modality (unimodal) 2

14 exceptional families:

E_{12}	$x^3 + y^7 + axy^5$	W_{12}	$x^4 + y^5 + ax^2y^3$
E_{13}	$x^3 + xy^5 + ay^8$	W_{13}	$x^4 + xy^4 + ay^6$
E_{14}	$x^3 + y^8 + axy^6$	Q_{10}	$x^3 + y^4 + yz^2 + axy^3$
Z_{11}	$x^3y + y^5 + axy^4$	Q_{11}	$x^3 + y^2z + xz^3 + az^5$
Z_{12}	$x^3y + xy^4 + ax^2y^3$	Q_{12}	$x^3 + y^5 + yz^2 + axy^4$
Z_{13}	$x^3y + y^6 + axy^5$	S_{11}	$x^4 + y^2z + xz^2 + ax^3z$
U_{12}	$x^3 + y^3 + z^4 + axyz^2$	S_{12}	$x^2y + y^2z + xz^3 + az^5$

Classification of low modality (bimodal)

Theorem

There are 8 infinite series and 14 exceptional families in the case $m = 2$.

4 series of singularities of corank 2:

Name	Normal form	Restrictions	Multiplicity μ
$J_{3,0}$	$x^3 + bx^2y^3 + y^9 + cxy^7$	$4b^3 + 27 \neq 0$	16
$J_{3,p}$	$x^3 + x^2y^3 + \mathbf{a}y^{9+p}$	$p > 0, a_0 \neq 0$	$16 + p$
$Z_{1,0}$	$x^3y + dx^2y^3 + cxy^6 + y^7$	$4d^3 + 27 \neq 0$	15
$Z_{1,p}$	$x^3y + x^2y^3 + \mathbf{a}y^{7+p}$	$p > 0, a_0 \neq 0$	$15 + p$
$W_{1,0}$	$x^4 + \mathbf{a}x^2y^3 + y^6$	$a_0^2 \neq 4$	15
$W_{1,p}$	$x^4 + x^2y^3 + \mathbf{a}y^{6+p}$	$p > 0, a_0 \neq 0$	$15 + p$
$W^{\#}_{1,2q-1}$	$(x^2 + y^3)^2 + \mathbf{a}xy^{4+q}$	$q > 0, a_0 \neq 0$	$15 + 2q - 1$
$W^{\#}_{1,2q}$	$(x^2 + y^3)^2 + \mathbf{a}x^2y^{3+q}$	$q > 0, a_0 \neq 0$	$15 + 2q$

4 series of singularities of corank 3:

Name	Normal form	Restrictions	Multiplicity μ
$Q_{2,0}$	$x^3 + yz^2 + \mathbf{a}x^2y^2 + xy^4$	$a_0^2 \neq 4$	14
$Q_{2,p}$	$x^3 + yz^2 + x^2y^2 + \mathbf{a}y^{6+p}$	$p > 0, a_0 \neq 0$	$14 + p$
$S_{1,0}$	$x^2z + yz^2 + y^5 + \mathbf{a}zy^3$	$a_0^2 \neq 4$	14
$S_{1,p}$	$x^2z + yz^2 + x^2y^2 + \mathbf{a}y^{5+p}$	$p > 0, a_0 \neq 0$	$14 + p$
$S^{\#}_{1,2q-1}$	$x^2z + yz^2 + zy^3 + \mathbf{a}xy^{3+q}$	$q > 0, a_0 \neq 0$	$14 + 2q - 1$
$S^{\#}_{1,2q}$	$x^2z + yz^2 + zy^3 + \mathbf{a}x^2y^{2+q}$	$q > 0, a_0 \neq 0$	$14 + 2q$
$U_{1,0}$	$x^3 + xz^2 + xy^3 + \mathbf{a}y^3z$	$a_0(a_0^2 + 1) \neq 0$	14
$U_{1,2q-1}$	$x^3 + xz^2 + xy^3 + \mathbf{a}y^{1+q}z^2$	$q > 0, a_0 \neq 0$	$14 + 2q - 1$
$U_{1,2q}$	$x^3 + xz^2 + xy^3 + \mathbf{a}y^{3+q}z$	$q > 0, a_0 \neq 0$	$14 + 2q$

14 exceptional families:

E_{18}	$x^3 + y^{10} + \mathbf{a}xy^7$	W_{17}	$x^4 + xy^5 + \mathbf{a}y^7$
E_{19}	$x^3 + xy^7 + \mathbf{a}y^{11}$	W_{18}	$x^4 + y^7 + \mathbf{a}x^2y^4$
E_{20}	$x^3 + y^{11} + \mathbf{a}xy^8$	Q_{16}	$x^3 + yz^2 + y^7 + \mathbf{a}xy^5$
Z_{17}	$x^3y + y^8 + \mathbf{a}xy^6$	Q_{17}	$x^3 + yz^2 + xy^5 + \mathbf{a}y^8$
Z_{18}	$x^3y + xy^6 + \mathbf{a}y^9$	Q_{18}	$x^3 + yz^2 + y^8 + \mathbf{a}xy^6$
Z_{19}	$x^3y + y^9 + \mathbf{a}xy^7$	S_{16}	$x^2z + yz^2 + xy^4 + \mathbf{a}y^6$
U_{16}	$x^3 + xz^2 + y^5 + \mathbf{a}x^2y^2$	S_{17}	$x^2z + yz^2 + y^6 + \mathbf{a}zy^4$

In the last three tables $\mathbf{a} = a_0 + a_1y$.

Connection to Klein singularity

There is a remarkable connection exists between the classification of simple singularities, that of regular polyhedra in three-dimensional space, and that of the Coxeter groups A_k, D_k, E_k.

In 1880, Klein employed polynomial equations coming from the Invariant theory of a finite subgroup of $SL(2, \mathbb{C})$ to solve the quintic equation. The invariant polynomials that appear in this way are singular at the origin. We will call the singularities arising from a quotient of $\mathbb{C}^2$ by a finite subgroup $SL(2, \mathbb{C})$ Kleinian singularities.

All the finite subgroups of SO(3) are exhausted by the following list:

1. the cyclic group $\mathbf{C}_n$ cyclic group of rotation;
2. the dihedral group $\mathbf{D}_{2n}$, isomorphic to the semidirect product of $\mathbf{C}_n$ and $\mathbf{C}_2$;
3. the isometric group of the tetrahedron $\mathbf{T}_{12}$;
4. the isometric group of the octahedron and cube $\mathbf{O}_{24}$;
5. the isometric group of the icosahedron and dodecahedron $\mathbf{I}_{60}$.

Connection to Klein singularity 2

Let Γ be a discrete subgroup of $SO(3)$. Consider its preimage $\Gamma^* \subset SU(2)$ under the 2-covering map $SU(2) \to SO(3)$.

All finite subgroups of $SU(2)$ are isomorphic to one of the following 5 groups: $\mathbf{C}_n$, $\mathbf{D}^*_{2n}$, $\mathbf{T}^*_{12}$, $\mathbf{O}^*_{24}$, $\mathbf{I}^*_{60}$.

The group Γ^* acts on $\mathbb{C}^2$ as a subgroup of $SU(2)$. Consider the algebra of polynomial invariants of this action of Γ^*,i.e. $\mathbb{C}[T_1, T_2]^{\Gamma^*}$. Klein showed that this algebra is generated by three invariants $x, y, z \in \mathbb{C}[T_1, T_2]$, which satisfy a single relation u. i.e. $\mathbb{C}[x, y, z]/u \cong \mathbb{C}[T_1, T_2]^{\Gamma^*}$. That relation u defines a hypersurface V in the space $\mathbb{C}^3$ with coordinates x, y, z. So V is naturally isomorphic to the orbit space of the action of Γ^* on $\mathbb{C}^2$ and has an isolated singular point at the origin.

Connection to Klein singularity 3

symmetric group	relation	singularity
$\mathbf{C}_n$	$x^n + yz = 0$	$A_{n-1},\ n \geq 2$
$\mathbf{D}^*_{2n}$	$x^2y - y^{n+1} + z^2 = 0$	$D_{n+2},\ n \geq 2$
$\mathbf{T}^*_{12}$	$x^3 + y^4 + z^2 = 0$	E_6
$\mathbf{O}^*_{24}$	$x^3 + xy^3 + z^2 = 0$	E_7
$\mathbf{I}^*_{60}$	$x^3 + y^5 + z^2 = 0$	E_8

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Milnor fibration main result 1

Theorem (Fibration theorem)

Let f be a non-constant polynomial in n variables $f : \mathbb{C}^n \to \mathbb{C}$. If $\mathbf{z_0}$ is any point of the complex hypersurface $V = f^{-1}(0)$ and if S_ϵ is a sufficiently small sphere centered at $\mathbf{z_0}$, let intersection be $K := S_\epsilon \cap V$, then the map $\phi : S_\epsilon \setminus K \to S^1$ given by

$$\phi(\mathbf{z}) = \frac{f(\mathbf{z})}{||f(\mathbf{z})||}$$

*from $S_\epsilon \setminus K$ to the unit circle is the projection map of a **smooth fiber bundle**. Each fiber $F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \setminus K$ is a smooth parallelizable $2(n-1)$-dimensional manifold. K is a $(2n-3)$-dimensional knotted manifold in sphere S^{2n-1} and is the boundary of the closure of F_θ.*

Milnor fibration main result 2

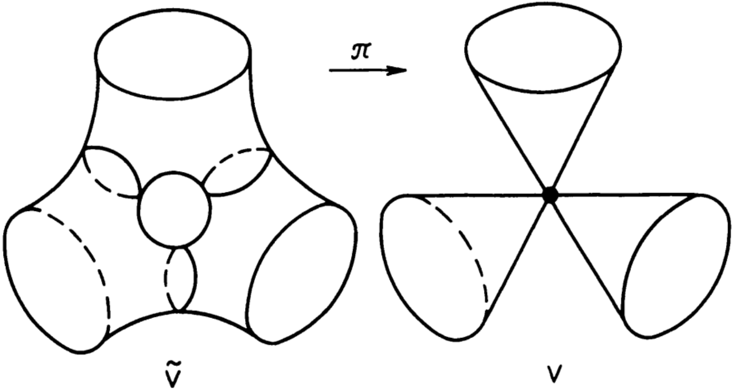
Theorem (Milnor)

*If $\mathbf{z_0}$ is an isolated critical point of f, then each fiber F_θ has the **homotopy type** of a bouquet of $n-1$-sphere $\bigvee^{\mu} S^{n-1}$ the number of spheres in this bouquet (i.e., the middle Betti number of F_θ), being strictly positive, this number is called **Milnor number** of a isolated critical point.*

The only nonzero reduced homology of F_0 is the middle homology $H_{n-1}(F_0, \mathbb{Z}) \cong \mathbb{Z}^\mu$. Since F_0 is a $(2n-2)$-dim manifold, so there is a bilinear intersection form $Q : H_{n-1}(F_0, \mathbb{Z}) \times H_{n-1}(F_0, \mathbb{Z}) \to \mathbb{Z}$. A singularity is said to be elliptic (respectively, parabolic, hyperbolic) if its quadratic form is positive definite (respectively, semidefinite, has negative index of inertia 1).

Theorem

The elliptic singularities are precisely the simple singularities A, D and E with positive definite. If choosing proper basis, one can write the intersection form matrix to be the standard Cartan matrix.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Exotic sphere first constructed by Milnor 1

For a differentiable oriented closed manifold M^8, Hirzebruch found a formula to represent the signature as the rational coefficients linear combination of Pontrjagin numbers: $\sigma(M^8) = \frac{7}{45}p_2[M^8] - \frac{1}{45}p_1^2[M^8]$, or $45\sigma + p_1^2 = 7p_2$.

Thom shows that any differentiable closed manifold Σ^7 is a boundary of a differentiable manifold M^8. If the boundary M^8 is a homology 7-sphere, then the signature still makes sense. Furthermore, the first Pontrjagin class, p_1, is well defined as an element of $H^4(M^8) \cong H^4(M^8, \partial M^8)$, and p_1^2 considered as an element of $H^8(M^8, \partial M^8)$, is non-zero, so $p_1^2[M^8]$ can be defined. If this boundary Σ^7 is actually diffeomorphic to standard sphere S^7. Then we can glue on an 8-ball to get a closed manifold. Hence we must have $45\sigma + p_1^2 \equiv 0 \; mod \; 7$.

Consider 3-sphere bundles over the 4-sphere with the rotation group SO(4) as structural group. The equivalence classes of such bundles are in one to one correspondence with elements of the homotopy groups $\pi_3(SO(4)) = \mathbb{Z}^2$, since $[S^4, BSO(4)] \cong [S^3, \Omega BSO(4)] \cong \pi_3(SO(4))$.

Exotic sphere first constructed by Milnor 2

For each $(h, j) \in \mathbb{Z}^2$, let $f_{hj} : S^3 \to SO(4)$ be defined by $f_{hj}(u) \cdot v = u^h v u^j$ using quaternion multiplication. Let $\Sigma^7 := E_{hj}$ denote the sphere bundle corresponding to f. Choosing proper (h, j) one can make the congruence equation fail.

There exists a differentiable function $f : \Sigma^7 \to \mathbb{R}$ having only two critical points. Furthermore these critical points are non-degenerate. So this Σ^7 is homeomorphic to S^7, Milnor found an exotic sphere.

Denote Θ_7 as the set of equivalence classes of oriented differentiable manifolds with the homotopy type of S^n (i.e., homotopy n-spheres), up to h-cobordism. By h-cobordism theory, if $n \geq 5$, all homotopy sphere is a homeomorphism sphere. Therefore, Θ_7 provide all diffeomorphism structures on 7-sphere.

How many diffeomorphism structures on 7-sphere

Theorem (Milnor, Kervaire)

Let $S_1,\ S_2$ be homotopy spheres of dimension $4m-1, (m>1)$ which bound s-parallelizable manifolds $M_1,\ M_2$. Then S_1 is h-cobordant to S_2 if and only if $\sigma(M_1) \equiv \sigma(M_2)\ mod\ \sigma_m$, where $\sigma_m = 2^{2m-1}(2^{2m-1}-1)\frac{B_m j_m a_m}{m}$. a_m is 1 if m is even or 2 if m is odd. B_m denotes the m-th Bernoulli number. j_m denotes the order of the cyclic group of $J(\pi_{4m-1}(SO)) \subset \pi_{4m-1}(S^0)$.

Theorem (Adams)

The value of j_m is precisely the denominator of $\frac{B_m}{4m}$.

How many diffeomorphism structures on 7-sphere

Corollary

If $n = 7$, then $m = 2$, a direct calculation shows that $\sigma_2 = 224$.

Theorem

Let M be an s-parallelizable $(2k-1)$-connected $4k$-manifold whose boundary is a homotopy sphere. Then the signature $\sigma(M)$ is a multiple of 8.

Theorem

Let $k > 1$ and $t \in \mathbb{Z}$. Then there exists an s-parallelizable $4k$-manifold M with boundary a homotopy sphere and signature $\sigma(M) = 8t$.

Theorem

There are precisely 28 differentiable structures on the 7-sphere.

Pham first studies the singularity of the function $f_{\mathbf{a}}(x_1, \cdots, x_n) = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$, here a is a vector $(a_1, \cdots, a_n)$. Then we can define the corresponding subvariety $K_{\mathbf{a}} = f_{\mathbf{a}}^{-1}(0) \cap S_\epsilon$ is a smooth $(2n-3)$-manifold. $K_{\mathbf{a}}$ is diffeomorphic to $K_{\mathbf{a}}(\delta) = f_{\mathbf{a}}^{-1}(\delta) \cap S_\epsilon$, $K_{\mathbf{a}}(\delta)$ is the boundary of smooth manifold $M_{\mathbf{a}} = f_{\mathbf{a}}^{-1}(\delta) \cap D_\epsilon$, D_ϵ is the ball with radius ϵ.

Theorem (Brieskorn)

Take $\mathbf{a} = (3, 6k-1, 2, 2, 2)$, then $K_{\mathbf{a}}$ is precisely the k-th exotic diffeomorphism sphere of the group Θ_7.

Theorem (Brieskorn)

If $n \geq 5$ is odd, then $K_{\mathbf{a}}(\delta)$ is a topological sphere, and the diffeomorphism type of $K_{\mathbf{a}}(\delta)$ is completely determined by the signature $\sigma_{\mathbf{a}} = \sigma(M_{\mathbf{a}})$. We have $\sigma_a = \sigma_a^+ - \sigma_a^-$, where σ_a^+ is determined to be the number of n-tuples of integers $(x_1, \cdots, x_n)$, with $0 < x_i < a_i$ such that

$$0 < \sum_{j=1}^{n} \frac{x_j}{a_j} < 1, \quad mod\ 2,$$

and σ_a^- is determined to be the number of n-tuples of integers $(x_1, \cdots, x_n)$, with $0 < x_i < a_i$ such that

$$-1 < \sum_{j=1}^{n} \frac{x_j}{a_j} < 0, \quad mod\ 2.$$

Brieskorn varieties and topological spheres

Example

Let us take n odd and $\mathbf{a} = (3, 6k-1, 2, \cdots, 2)$. Then $K_{\mathbf{a}}(\delta)$ is a topological sphere, and $\sigma_{\mathbf{a}} = (-1)^{\frac{n-1}{2}} 8k$.

Corollary

Let $n = 5$, $\mathbf{a} = (3, 6k-1, 2, 2, 2)$, then $\sigma_{\mathbf{a}} = 8k$, then $K_{\mathbf{a}}$ is precisely the k-th exotic diffeomorphism sphere of the group Θ_7.

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