Introduction

Adapted Homology Theory

Classification of Adams Spectral Sequence

Prestable Freyd Envelope

What To Do Next?

What is $D(\mathbb{S})$?

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What To Do Next?

About me

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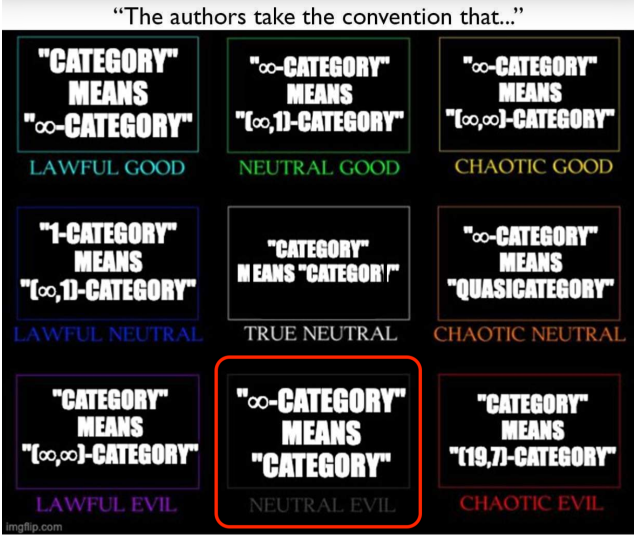
"The authors take the convention that..."
"CATEGORY" MEANS "∞-CATEGORY"
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TRUE NEUTRAL
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Overview

1. Introduction
2. Adapted Homology Theory
3. Classification of Adams Spectral Sequence
4. Prestable Freyd Envelope
5. What To Do Next?

Algebraic Topology

Question:Can we find a topological object realizing the given algebraic data?

Algebraic Topology

Our first attempt

Goerss-Hopkins Theory

Answer: Vanishing Goerss-Hopkins obstructions

$$\{\text{The topological guys}\} \simeq \mathcal{M}_\infty \to \cdots \to \mathcal{M}_0 \simeq \{\text{The algebraic guys}\}$$

Fact: this tower is an Postnikov tower in a suitable context.

Homology Theory

Definition

Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{A}$ be an abelian category. We say a functor $H\colon \mathcal{C} \to \mathcal{A}$ is is *homological* if

- H is additive and
- if
$$c \to d \to e$$
is a cofibre sequence in $\mathcal{C}$, then
$$H(c) \to H(d) \to H(e)$$
is exact.

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Homology Theory

Definition

A *local grading* on an ∞-category $\mathcal{C}$ is an autoequivalence

$$[1]_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}.$$

A *locally graded* ∞-category is a pair of an ∞-category and a local grading.

Homology Theory

Definition

We say a functor $H\colon \mathcal{C} \to \mathcal{A}$ of locally graded ∞-categories is a *homology theory* if its underlying functor is homological.

Homology Theory

Example

Let $\mathcal{A} = \mathsf{Vect}(\mathbb{F}_p)$ be the abelian category of graded vector spaces. Then, the mod p homology functor

$$H_*(-, \mathbb{F}_p) \colon \mathcal{S}p \to \mathsf{Vect}(\mathbb{F}_p)$$

is canonically a homology theory.

Adaptedness

Definition

We say a homology theory $H\colon \mathcal{C} \to \mathcal{A}$ has *lifts for injectives* if

- $\mathcal{A}$ has enough injectives and
- any injective $i \in \mathcal{A}$ lifts to some $i_{\mathcal{C}}$.

Adaptedness

Example

Consider $H_*(-, \mathbb{F}_p)\colon \mathcal{S}p \to \mathsf{Vect}(\mathbb{F}_p)$. In $\mathsf{Vect}(\mathbb{F}_p)$, $\mathbb{F}_p$ is injective, and we have

$$\mathrm{Hom}_{\mathbb{F}_p}(H_*(X, \mathbb{F}_p), \mathbb{F}_p) \simeq H^*(X, \mathbb{F}_p) \simeq [X, H\mathbb{F}_p],$$

Thus, the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ is the associated injective lift.

Warning

Warning

We have seen that the injective associated to $\mathbb{F}_p$ via the homology theory $H_*(-, \mathbb{F}_p)\colon \mathcal{S}p \to \mathsf{Vect}(\mathbb{F}_p)$ is given by the Eilenberg-MacLane spectrum $H\mathbb{F}_p$.

However, $H_*(H\mathbb{F}_p, \mathbb{F}_p) \simeq A_*$ is the dual Steenrod algebra, and the comparison map $A_* \to \mathbb{F}_p$ is not an isomorphism.

In light of the above warning, we make the following definition.

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In light of the above warning, we make the following definition.

Adaptedness

Definition

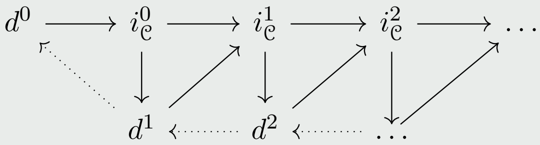
We say a homology theory $H \colon \mathcal{C} \to \mathcal{A}$ is *adapted* if:

- $\mathcal{A}$ has enough injectives,
- any injective $i \in \mathcal{A}$ lifts to an associated injective $i_{\mathcal{C}}$ (i.e. H has lifts for injectives) and,
- the structure map $H(i_{\mathcal{C}}) \to i$ is an isomorphism.

Adams Spectral Sequence

Construction (Adams spectral sequence)

Let $H\colon \mathcal{C} \to \mathcal{A}$ be an adapted homology theory and $d \in \mathcal{C}$ be an object. Writing $d = d^0$, we can find an embedding $H(d^0) \hookrightarrow i^0$ into an injective, which determines a map $d \to i^0_{\mathcal{C}}$ into a corresponding associated injective object of $\mathcal{C}$. Proceeding inductively by setting $d^{i+1} = cofib(d^i \to i^i_{\mathcal{C}})$, we construct an *Adams resolution* of the form



Adams Spectral Sequence

Construction (Adams spectral sequence)

Applying H, we obtain an injective resolution of $H(d) = H(d^0)$. Applying $[c, -]_*$ for some other object $c \in \mathcal{C}$, we obtain a spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H(c), H(d)),$$

This is the *H-Adams spectral sequence*, and in favourable cases it converges to $[c, d]_*$ (or a suitable completion).

Adams Spectral Sequence

Example

The mod p homology $H_*(X, \mathbb{F}_p)$ of a spectrum X has a canonical structure of a comodule over the dual Steenrod algebra A_*. Thus, the previously considered homology theory factors as

$$\mathcal{S}p \to \mathcal{C}omod(A_*) \to \mathsf{Vect}(\mathbb{F}_p),$$

where $\mathcal{C}omod(A_*)$ is the category of comodules over A_* and the second functor is the forgetful one.

Example

In fact, one can show that

$$H_*(-, \mathbb{F}_p)\colon \mathcal{S}p \to \mathcal{C}omod(A_*)$$

is an adapted homology theory.

Classical Freyd Envelope

Definition

The *Freyd envelope* of an additive ∞-category $\mathcal{C}$, denoted by $A(\mathcal{C})$, is the full subcategory of $\mathrm{Fun}_{\Sigma}(\mathcal{C}^{op}, \mathcal{A}b)$ spanned by finitely presented presheaves.

Remarks

- $A(\mathcal{C})$ only depends on its homotopy category.
- The discrete Yoneda embeddding $y\colon \mathcal{C} \to A(\mathcal{C})$ is usually far from being fully faithful, but $y\colon h\mathcal{C} \to A(\mathcal{C})$ is fully faithful.

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Universal Property 1

By definition, the Freyd envelope is the smallest subcategory of the category $\mathrm{Fun}_{\Sigma}(\mathcal{C}^{op}, \mathcal{A}b)$ of all additive presheaves which contains all representables and is closed under isomorphisms and under finite colimits.

Universal Property 1

Using this description, one can show that the Freyd envelope $A(\mathcal{C})$ enjoys the following universal property:

- if $\mathcal{B}$ is an additive 1-category which admits finite colimits, any additive functor $\mathcal{C} \to \mathcal{B}$ uniquely extends to a *right exact* additive functor $A(\mathcal{C}) \to \mathcal{B}$.

In other words, the classical Freyd envelope $A(\mathcal{C})$ is obtained from the homotopy category $h\mathcal{C}$ by formally adjoining cokernels. In fact, as cokernels are reflexive coequalizers, one can show that $A(\mathcal{C})$ is obtained from $h\mathcal{C}$ by formally adjoining reflexive coequalizers.

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Abelian Subcategory

Proposition

Let $\mathcal{C}$ be an additive ∞-category with **finite limits**. *Then, the Freyd envelope $A(\mathcal{C})$ is an abelian subcategory of* $\text{Fun}_{\Sigma}(\mathcal{C}^{op}, \mathcal{A}b)$; *that is, it is closed under extensions, finite limits and colimits.*

In fact, one can show that $A(\mathcal{C})$ can be described as the smallest subcategory of $\text{Fun}_{\Sigma}(\mathcal{C}^{op}, \mathcal{S}et)$ containing all the representables and closed under finite colimits.

The Preferred Local Grading

Definition

Let $\mathcal{C}$ be a locally graded additive ∞-category. Then, the *induced local grading* on the Freyd envelope $A(\mathcal{C})$ is defined by

$$(X[1])(c) := X(c[-1]).$$

Universal Property 2

Theorem (Freyd)

Let $\mathcal{C}$ be a stable ∞-category. Then, the the functor $y\colon \mathcal{C} \to A(\mathcal{C})$ is homological. Moreover, it is universal in the following sense: for any homological functor $H\colon \mathcal{C} \to \mathcal{A}$, there is an essentially unique exact functor $L\colon A(\mathcal{C}) \to \mathcal{A}$ of abelian categories such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad H \quad} & \mathcal{A} \\ & {\scriptstyle y}\searrow \quad \nearrow{\scriptstyle \exists! L} & \\ & \mathcal{A}(\mathcal{C}) & \end{array} .$$

$A(\mathcal{C})$ is usually not presentable

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Warning

Even if $\mathcal{C}$ is a presentable ∞-category, $A(\mathcal{C})$ will not be presentable except in the most trivial cases!

Characterization of Adapted Homology Theories

Lemma

Let $\mathcal{C}$ be an **idempotent complete** *stable ∞-category and $H\colon \mathcal{C} \to \mathcal{A}$ a homology theory such that $\mathcal{A}$ has enough injectives. Then, the following are equivalent:*

- *every injective of $\mathcal{A}$ has a lift in $\mathcal{C}$,*
- *induced functor $L\colon A(\mathcal{C}) \to \mathcal{A}$ has a right adjoint.*

Characterization of Adapted Homology Theories

Theorem (Characterization of adapted homology theories)

Let $\mathcal{C}$ be an **idempotent complete** *stable ∞-category and $H : \mathcal{C} \to \mathcal{A}$ a homology theory such that $\mathcal{A}$ has enough injectives. Then, the following conditions are equivalent:*

- *H is adapted,*
- *the induced functor $L : A(\mathcal{C}) \to \mathcal{A}$ has a fully faithful right adjoint,*

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Epimorphism Class

Definition

Let $\mathcal{C}$ be a stable ∞-category. We say a class of arrows $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is an **epimorphism class** if:

- all equivalences are in $\mathcal{E}$,
- for any pair f, g of composable arrows, if $f, g \in \mathcal{E}$ then also $g \circ f \in \mathcal{E}$ and if $g \circ f \in \mathcal{E}$ then also $g \in \mathcal{E}$,
- $\mathcal{E}$ is stable under pullbacks along arbitrary maps in $\mathcal{C}$,
- an arrow $f : c \to d$ belongs to $\mathcal{E}$ if and only if $\Sigma f : \Sigma c \to \Sigma d$ does.

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Epimorphism Class

Definition

Recall that in an ∞-category $\mathcal{C}$, a morphism $c \to d$ is called an effective epimorphism if the Čech nerve

$$\dots \ \substack{\to\\\to\\\to} \ c \times_d c \rightrightarrows c \to d$$

is a colimit diagram.

Epimorphism Class

Warning

- In an abelian category $\mathcal{A}$ is an effective epimorphism if and only if it is an epimorphism.
- in a stable ∞-category $\mathcal{C}$, *any* arrow $c \to d$ is an effective epimorphism.

Warning: There is **no** natrual choice of the class of "epimorphisms" in a stable ∞-category $\mathcal{C}$.

Major Examples of Epimorphism Classes

Example

The class of *split surjections*; that is, those maps $c \to d$ which are equivalent to a projection onto a direct summand, is an epimorphism class.

Example

Let $H\colon \mathcal{C} \to \mathcal{A}$ be a homology theory. Then, the class of H-epimorphisms; that is, those maps $c \to d$ such that $H(c) \to H(d)$ is an epimorphism, is an epimorphism class on $\mathcal{C}$.

Example

Let $L\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories and let $\mathcal{E}$ be an epimorphism class on $\mathcal{D}$. Then, $L^{-1}(\mathcal{E})$ is an epimorphism class in $\mathcal{C}$.

Epimorphism Class

Definition

Let $\mathcal{E}$ be an epimorphism class on a stable ∞-category $\mathcal{C}$. We will say an arrow $c \to d$ is $\mathcal{E}$-*monic* if the canonical map $d \to \mathrm{cofib}(c \to d)$ is in $\mathcal{E}$.

Definition

Let $\mathcal{E}$ be an epimorphism class. We say an object $i \in \mathcal{C}$ is $\mathcal{E}$-*injective* if for every $c \to d$ in $\mathcal{E}$, the induced map

$$[d, i] \to [c, i]$$

is a monomorphism of abelian groups.

Epimorphism Class

Definition

Let $\mathcal{E}$ be an epimorphism class. We say $\mathcal{C}$ *has enough $\mathcal{E}$-injectives* if for every $c \in \mathcal{C}$ there exists an $\mathcal{E}$-monic map $c \to i$ into an $\mathcal{E}$-injective.

Classification of Adams Spectral Sequences

Theorem (Classification of Adams spectral sequences)

The following two sets of data one can associate to an idempotent-complete stable ∞-category $\mathcal{C}$ are equivalent:

1) *an epimorphism class $\mathcal{E}$ of morphisms such that $\mathcal{C}$ has enough $\mathcal{E}$-injectives,*
2) *Bousfield localizations $A(\mathcal{C}) \to \mathcal{A}$ of such that the Gabriel quotient $\mathcal{A}$ has enough injectives and*
3) *adapted homology theories $H: \mathcal{C} \to \mathcal{A}$.*

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Two ways to generalize Freyd Envelope

- We have mentioned that the classical Freyd envelope $A(\mathcal{C})$ is obtained from $h\mathcal{C}$ by formally adjoining reflexive coequalizers.
- The ∞-categorical analogue of a reflexive coequalizers is given by a geometric realization of a simplicial object.
- To obtain an ∞-categorical analogue of the Freyd envelope, we should enlarge a given $\mathcal{C}$ by freely adjoining *geometric realizations*.

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Two ways to generalize Freyd Envelope

- Alternatively, if $\mathcal{C}$ admits finite limits, then $A(\mathcal{C})$ can be described as the smallest subcategory of $\mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathcal{S}et)$ containing all the representables and closed under finite colimits.
- This suggests a different generalization of the Freyd envelope, where we take the same definition, but replace sets by the ∞-category of spaces.

Almost Perfect Presheaves

Definition

Let $\mathcal{C}$ be an additive ∞-category. We say a product-preserving presheaf $X : \mathcal{C}^{op} \to \mathcal{S}$ of spaces is *almost perfect* if it belongs to the smallest subcategory of $\mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathcal{S})$ containing the representables and closed under geometric realizations.

Definition

The *prestable Freyd envelope* $A_\infty(\mathcal{C})$ is the ∞-category of almost perfect presheaves on $\mathcal{C}$. It is an additive subcategory of $\mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathcal{S})$.

Almost Perfect Presheaves

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The Restricted Yoneda Embedding

Notation

The Yoneda embedding of $\mathcal{C}$ factors through almost perfect presheaves, and we will denote this factorization by $\nu\colon \mathcal{C} \to A_\infty(\mathcal{C})$. Explicitly, we have

$$\nu(c)(d) := \mathsf{Map}_{\mathcal{C}}(d, c).$$

Universal Property 1

Remark

By construction, the ∞-category $A_\infty(\mathcal{C})$, has the following universal property:

for any ∞-category $\mathcal{D}$ admitting geometric realizations, restriction along the Yoneda embedding gives an equivalence

$$\mathrm{Fun}_\sigma(A_\infty(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

between geometric-realization preserving functors out of $A_\infty(\mathcal{C})$, and all functors out of $\mathcal{C}$. The inverse to the above equivalence is given by left Kan extension.

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Characterization of Almost Perfect Presheaves

Proposition

Let $\mathcal{C}$ be an additive ∞-category which admits finite limits. Then, the following conditions on a product-preserving presheaf $X : \mathcal{C}^{op} \to \mathcal{S}$ are equivalent:

- $X \simeq |\nu(c_\bullet)|$ *for a simplicial diagram of representables,*
- X *is almost perfect,*
- $\pi_k X$ *is finitely presented presheaf of abelian groups; that is,* $\pi_k X \in A(\mathcal{C})$*, for every* $k \geq 0$*.*

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Characterization of Almost Perfect Presheaves

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Xiansheng Li

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Corollary

Suppose that $\mathcal{C}$ is an additive ∞-category which admits finite limits. Then, the prestable Freyd envelope $A_\infty(\mathcal{C})$ is closed under finite limits, finite colimits, extensions and Postnikov truncations in $\mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathcal{S})$. In particular, it is prestable.

Prestable ∞-category

Definition

A ∞-category $\mathcal{C}$ is called prestable if the following hold:

- $\mathcal{C}$ is pointed and admits finite colimits,
- the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully failthful and
- for any arrow of the form $d \to \Sigma c$, where $c, d \in \mathcal{C}$, there exists a bicartesian square

$$\begin{array}{ccc} e & \longrightarrow & d \\ \downarrow & \lrcorner \quad \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma c \end{array}$$

If you do not familiar with this, think of it as the connective part of your favorite stable ∞-category(precise when $\mathcal{C}$ admits finite limits).

t-structure Homotopy Groups

Notation

If $\mathcal{C}$ is prestable, the *heart* is $\mathcal{C}^{\heartsuit} := \tau_{\leq 0}\mathcal{C}$, the subcategory of discrete objects. If $\mathcal{C}$ has finite limits, then the inclusion of the heart admits a left adjoint denoted by

$$\pi_0 \colon \mathcal{C} \to \mathcal{C}^{\heartsuit}.$$

Using the heart and π_0 we define the higher *homotopy groups* by setting

$$\pi_k(c) := \pi_0(\Omega^k c).$$

These correspond to the usual t-structure homotopy groups on the stabilization, although in the prestable setting they are necessarily non-negatively graded.

t-structure Homotopy Groups

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Using the heart and π_0 we define the higher *homotopy groups* by setting

$$\pi_k(c) := \pi_0(\Omega^k c).$$

These correspond to the usual t-structure homotopy groups on the stabilization, although in the prestable setting they are necessarily non-negatively graded.

t-structure Homotopy Groups

Example

Let $\mathcal{C}$ be an additive ∞-category which admits finite limits. A presheaf $X \in A_\infty(\mathcal{C})$ is discrete if and only if it is discrete as an object of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$; that is, when it is valued in sets.
It follows that we have an equivalence

$$A_\infty(\mathcal{C})^\heartsuit \simeq A(\mathcal{C}).$$

In terms of this equivalence, the t-structure homotopy groups coincide with the pointwise ones in the sense that

$$(\pi_k X)(c) \simeq \pi_k X(c).$$

Local Grading

Remark

Using the universal property of $A_\infty(\mathcal{C})$, we see that if $\mathcal{C}$ is locally graded, then $A_\infty(\mathcal{C})$ inherits an essentially unique local grading such that ν admits a structure of a locally graded functor.
When $\mathcal{C}$ is stable and locally graded through suspension, we have

$$(X[1])(c) := X(\Sigma^{-1}c);$$

Perfect Freyd Envelope

Definition

Let $\mathcal{C}$ be an additive ∞-category. Then, an additive presheaf $X \in \mathrm{Fun}_{\Sigma}(\mathcal{C}^{op}, \mathcal{S})$ is *perfect* if it belongs to the smallest subcategory containing all representables and closed under finite colimits.

Definition

The *perfect prestable Freyd envelope* is the ∞-category $A^{\omega}_{\infty}(\mathcal{C})$ of perfect presheaves.

Universal Property

Remark

By construction, $A^{\omega}_{\infty}(\mathcal{C})$ has the following universal property: any additive functor $\mathcal{C} \to \mathcal{D}$ into an additive ∞-category with finite colimits extends uniquely to a right exact functor $A^{\omega}_{\infty}(\mathcal{C}) \to \mathcal{D}$.

Perfect Preseaves are Almost Perfect

Lemma

Let $\mathcal{C}$ be an additive ∞-category. Then, any perfect presheaf X is almost perfect; that is, $A_\infty^\omega(\mathcal{C})$ is naturally a subcategory of $A_\infty(\mathcal{C})$.

One-word Proof.

In an additive ∞-category, the cofiber of $d \to c$ is equivalent to the geometric realization of the bar construction

$$\cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} d \oplus d \oplus c \rightrightarrows d \oplus c \to c.$$

□

Finite Limits of Perfect Presheaves

Theorem

Let $\mathcal{C}$ be an additive ∞-category with finite limits. Then, the perfect prestable Freyd envelope $A^{\omega}_{\infty}(\mathcal{C})$ is closed under finite limits in additive presheaves of spaces. In particular, it is a prestable ∞-category with finite limits.

Prestable Enhancement

Definition

Let $\mathcal{C}$ be a stable ∞-category. We say a functor $\mathcal{H}\colon \mathcal{C} \to \mathcal{D}$ into a prestable ∞-category $\mathcal{D}$ with finite limits is a *prestable enhancement* to a homological functor when

- $\mathcal{H}$ is additive and
- $\mathcal{H}$ is left exact.

Prestable Enhancement

Example

The functor $\nu\colon \mathcal{C} \to A_\infty(\mathcal{C})$ is a prestable enhancement compatible with the local grading, as the Yoneda embedding into presheaves preserves all limits.

Universal Property 2

Theorem

Let $\mathcal{H} \colon \mathcal{C} \to \mathcal{D}$ be a prestable enhancement to a homological functor and suppose that $\mathcal{D}$ admits geometric realizations. Then, there exists an essentially unique exact, geometric realization-preserving functor $\mathcal{L} \colon A_\infty(\mathcal{C}) \to \mathcal{D}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad \mathcal{H} \quad} & \mathcal{D} \\ & {\scriptstyle \nu} \searrow \qquad \nearrow {\scriptstyle \exists ! \mathcal{L}} & \\ & \mathcal{A}_\infty(\mathcal{C}) & \end{array} .$$

Universal Property 2'

Theorem

Suppose that $\mathcal{C}$ is stable and let $\mathcal{H}\colon \mathcal{C} \to \mathcal{D}$ be a prestable enhancement to a homological functor. Then, there exists an essentially unique exact functor $\mathcal{L}\colon A^{\omega}_{\infty}(\mathcal{C}) \to \mathcal{D}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\mathcal{H}\quad} & \mathcal{D} \\ & {\scriptstyle \nu}\searrow \qquad \nearrow{\scriptstyle \exists ! \mathcal{L}} & \\ & A^{\omega}_{\infty}(\mathcal{C}) & \end{array} .$$

Thread Structure

Construction

Since ν is a functor, for any $c \in \mathcal{C}$ we have a canonical map

$$\Sigma \nu c \to \nu(\Sigma c) \simeq \nu(c)[1].$$

This is in fact a natural transformation of functors $\mathcal{C} \to A_\infty(\mathcal{C})$, and by left Kan extension we obtain a natural transformation

$$\tau \colon \Sigma X \to X[1]$$

of geometric realization-preserving endofunctors of the prestable Freyd envelope.

Thread Structure

Definition

The *canonical thread structure* on $A_\infty(\mathcal{C})$ is the above natural transformation

$$\tau\colon \Sigma X \to X[1].$$

Cofiber of τ^n

Notation

Let $\pi(n)\colon \mathcal{C} \to h_n\mathcal{C}$ be the projection onto the n-th homotopy category. We then have an adjunction

$$\pi(n)^* \dashv \pi(n)_*\colon A_\infty(\mathcal{C}) \rightleftarrows A_\infty(h_n\mathcal{C})$$

between the ∞-categories of almost perfect presheaves given by left Kan extension and restriction.

Cofiber of τ^n

Proposition

For any $n \geq 1$, the map τ^n into an almost perfect X fits into a canonical cofiber sequence

$$\Sigma^n X[-n] \to X \to \pi(n)_* \pi(n)^* X$$

Cofiber of τ^n

Notation

We will write

$$C\tau^n \otimes - := \pi(n)_* \pi(n)^*$$

for the monad associated to the adjunction $\pi(n)^* \dashv \pi(n)_*$. We will refer to modules over the monad $C\tau^n$ as $C\tau^n$-*modules*.

Cofiber of τ^n

Warning

Beware that this notation is potentially abusive, as unless $\mathcal{C}$ is monoidal, there is no natural notion of a tensor product of almost perfect presheaves.

However, we can treat $C\tau^n$ as if they were commtative algebras in the following sense

Lemma (Linearity of $C\tau^n$)

If X is a module over the monad $C\tau^n \otimes -$, then the map

$$\tau : \Sigma X[-1] \to X$$

has a canonical lift to a morphism of modules.

Cofiber of τ^n

Remark (Passing to perfect presheaves)

Note that since perfect presheaves are stable under cofibers. This implies that if $X \in A^{\omega}_{\infty}(\mathcal{C})$ is perfect, so is $C\tau^n \otimes X$ for any $n \geq 1$. In particular, the monad $C\tau^n$ restricts to a monad on the perfect prestable Freyd envelope.

Cofiber of τ^n

One part of the importance of the monad $C\tau^n$ is that it allows one to recover the Freyd envelopes of the homotopy categories, as the following shows.

Proposition

The adjunction $\pi(n)^* \dashv \pi(n)_* \colon A_\infty(\mathcal{C}) \rightleftarrows A_\infty(h_n\mathcal{C})$ is monadic; that is, it induces an equivalence

$$\mathcal{M}od_{C\tau^n \otimes -}(A_\infty(\mathcal{C})) \simeq A_\infty(h_n\mathcal{C})$$

with modules over the monad $C\tau^n$.

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Cofiber of τ^n

Proposition

The forgetful functor

$$\mathcal{M}od_{C\tau^n \otimes -}(A_\infty(\mathcal{C})) \to A_\infty(\mathcal{C})$$

is exact and induces an equivalence

$$\tau_{\leq n-1}(\mathcal{M}od_{C\tau^n \otimes -}(A_\infty(\mathcal{C}))) \simeq \tau_{\leq n-1} A_\infty(\mathcal{C})$$

between the subcategories of $(n-1)$-truncated objects.

τ-inversion

Construction (τ-inversion)

Let $\mathcal{C}$ be a stable ∞-category. Then, the left Kan extension provides a unique exact extension

$$\tau^{-1} \colon A^{\omega}_{\infty}(\mathcal{C}) \to \mathcal{C}$$

which we will call the τ*-inversion functor*. In the case when $\mathcal{C}$ admits geometric realizations, the analogous construction also applies to the almost perfect presheaves.

τ-inversion

Introduction

Adapted Homology Theory

Classification of Adams Spectral Sequence

Prestable Freyd Envelope

What To Do Next?

Construction (τ-inversion)

Let $\mathcal{C}$ be a stable ∞-category. Then, the left Kan extension provides a unique exact extension

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which we will call the *τ-inversion functor*. In the case when $\mathcal{C}$ admits geometric realizations, the analogous construction also applies to the almost perfect presheaves.

Characterization of perfect presheaves

Lemma

Let $\mathcal{C}$ be a stable ∞-category. Then, the following are equivalent for an additive presheaf $X \in \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathcal{S})$:

- *X is perfect,*
- *$\pi_k X$ is finitely presented for each $k \geq 0$, and there exists an N such that for any $k \geq N$, $\pi_k X$ is a representable discrete presheaf and $\tau \colon \pi_k X[-1] \to \pi_{k+1}(X)$ is an isomorphism.*

Set Theoretic issues

It will be convenient to argue in a context where we know that sheafification exists.

Notation

Let $\mathcal{C}$ be an additive ∞-category. Recall that $A_\infty(\mathcal{C})$ is by definition a subcategory of presheaves $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, choosing a larger universe, we can embed the latter into the ∞-category

$$\widehat{P}_\Sigma(\mathcal{C}) := \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \widehat{\mathcal{S}})$$

of product-preserving presheaves valued in large spaces. Relative to this larger universe, $\mathcal{C}$ is small, and so by standard results there exists a sheafification functor which we will denote by $L\colon \widehat{P}_\Sigma(\mathcal{C}) \to \widehat{P}_\Sigma(\mathcal{C})$.

H-epimorphism topology

Notation

We fix a stable ∞-category equipped with a choice of an adapted homology theory $H\colon \mathcal{C} \to \mathcal{A}$. We recall that by the characterization of adapted homology theories, the induced functor $A(\mathcal{C}) \to \mathcal{A}$ presents the latter as a bousfield localization whose kernel we will denote by K.

H-epimorphism topology

Definition

The *H-epimorphism topology* is a Grothendieck pretopology on $\mathcal{C}$ in which a family of maps $\{c_i \to d\}$ is a covering if it consists of a single map which is an H-epimorphism. For formal reasons, we have a sheafification functor

$$L\colon A^{\omega}_{\infty}(\mathcal{C}) \to A^{\omega}_{\infty}(\mathcal{C}).$$

Perfect Sheaves

Proposition

Let $X \in A^{\omega}_{\infty}(\mathcal{C})$ be a perfect presheaf. Then, the sheafification LX is perfect.

Here we take $\mathcal{C}$ to be either an abelian category equipped with the epimorphism topology or an stable ∞-category equipped with the H-epimorphism topology.
Restricting to perfect objects, we get rid of the larger universe!

Bounded Derived Category

Definition

Let $\mathcal{A}$ be an abelian category with enough injectives. The *bounded (connective) derived ∞-category* of $\mathcal{A}$, denoted by $\mathcal{D}^b(\mathcal{A})$, is the ∞-category of perfect sheaves on $\mathcal{A}$ with respect to the epimorphism topology.

Bounded Derived Category

The above definition is reasonable, as it satisfies the following universal property:

Theorem

Let $\mathcal{D}$ be a prestable ∞-category with finite limits. Then, left Kan extension gives an equivalence between the following data:

- *exact functors $\mathcal{A} \to \mathcal{D}^{\heartsuit}$ of abelian categories and*
- *exact functors $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}$ of prestable ∞-categories with finite limits.*

Perfect Derived Category

Definition

Let $H : \mathcal{C} \to \mathcal{A}$ be an adapted homology theory. Then, the perfect derived ∞-category of $\mathcal{C}$ relative to H is given by

$$\mathcal{D}^{\omega}(\mathcal{C}) := A_{\infty}^{\omega,sh}(\mathcal{C}),$$

the ∞-category of perfect sheaves on $\mathcal{C}$ with respect to the H-epimorphism topology.

Properties of Perfect Derived Categories

Proposition

We have that

1) $L\colon A^{\omega}_{\infty}(\mathcal{C}) \to \mathcal{D}^{\omega}(\mathcal{C})$ *is an exact localization compatible with local grading, in particular* $\mathcal{D}^{\omega}(\mathcal{C})$ *is a locally graded, prestable* ∞*-category with finite limits.*
2) *We have a canonical equivalence* $\mathcal{D}^{\omega}(\mathcal{C})^{\heartsuit} \simeq \mathcal{A}$.
3) $\nu\colon \mathcal{C} \to A^{\omega}_{\infty}(\mathcal{C})$ *factors through* $\mathcal{D}^{\omega}(\mathcal{C})$.
4) *A cofibre sequence* $c \to d \to e$ *in* $\mathcal{C}$ *induces a cofibre sequence* $\nu(c) \to \nu(d) \to \nu(e)$ *if and only if* $H(d) \to H(e)$ *is surjective.*

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Properties of Perfect Derived Categories

Theorem (Universal property of the perfect derived ∞-category)

Let $H\colon \mathcal{C} \to \mathcal{A}$ be an adapted homology theory, $\mathcal{D}^{\omega}(\mathcal{C})$ the corresponding perfect derived ∞-category and $\mathcal{D}$ a prestable ∞-category with finite limits. Then, left Kan extension along $\nu\colon \mathcal{C} \to \mathcal{D}^{\omega}(\mathcal{C})$ induces an equivalence between the two following collections of data:

1) *exact functors $G\colon \mathcal{D}^{\omega}(\mathcal{C}) \to \mathcal{D}$ of prestable ∞-categories and*
2) *exact functors $G_0\colon \mathcal{A} \to \mathcal{D}^{\heartsuit}$ of abelian categories together with a prestable enhancement of $G_0 \circ H$.*

The Two "Fibers"

Theorem

The right adjoint $H_* \colon \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^\omega(\mathcal{C})$ *has a canonical lift to* $C\tau$*-modules and induces an equivalence*

$$\mathcal{M}od_{C\tau\otimes -}(\mathcal{D}^\omega(\mathcal{C})) \simeq \mathcal{D}^b(\mathcal{A})$$

between $C\tau$*-modules whose underlying sheaf is perfect and the bounded derived* ∞*-category of* $\mathcal{A}$*.*

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$$\mathcal{C} \xleftarrow{\ \tau^{-1}\ } D^\omega(\mathcal{C}) \xrightarrow{\ C\tau\otimes -\ } D^b(\mathcal{A})$$

Further Topics

further topics

- Postnikov tower and Adams resolution,
- three variants of $D^{\omega}(\mathcal{C})$ (for the grothendieck abelian case),
- Goerss-Hopkins tower and algebraicity.

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Thank you!