Introduction to intersection homology. Ref: Singular intersection homology, Greg Friedman

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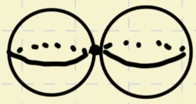
[Exp] Poincare duality $\overset{\text{Corollary}}{\Rightarrow}$ $\dim_{\mathbb{C}} H_i(X;\mathbb{C}) = \dim_{\mathbb{C}} H_{n-i}(X;\mathbb{C})$

$Y =$  $S^2 \vee S^2$ $\quad H_0(Y;\mathbb{C}) = \mathbb{C}$

$H_2(Y;\mathbb{C}) = \mathbb{C}\oplus\mathbb{C}$

singular space ⟵ Intersection homology

Outline:

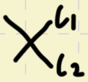
GM intersection homology (NOT right)
$\begin{cases} \text{Simplicial intersection homomology} \\ \text{PL intersection homology} \\ \text{singular intersection homology} \end{cases}$ + Non GM intersection homology

• [Def] (filtered space) A filtered space is a Hausdorff topo space X together with a seq. of closed subspaces

$$X = X^n \supseteq X^{n-1} \supseteq \cdots \supseteq X^{-1} = \phi$$

X^i: i-th skeleton; connected component of $X^i - X^{i-1}$: stratum; $X^n - X^{n-1}$: regular stratum; index i: formal dimension; $X^{n-1} =: \Sigma_X$

[Rmk] 1. X^i is always i-dimension singularities

$\mathbb{R}^2$  $\quad X^2 = \mathbb{R}^2 \supseteq X^1 = \underbrace{l_1 \cup l_2}_{\text{1-dim sing.}} \supseteq X^0 = \underbrace{l_1 \cap l_2}_{\text{0-dim sing.}} \supseteq X^{-1} = \phi$

2. Formal dimension can not equal to topo dim.

e.g (subspace filtration) (Analog to subspace topology)

$Y \subseteq X$, $X^n \supseteq X^{n-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \phi$

we define $Y^i = X^i \cap Y \rightsquigarrow Y^n \supseteq Y^{n-1} \supseteq \cdots \supseteq Y^{-1} = \phi$.

$X = S^2 \vee_* S^1$. $Y = S^1$. $\quad X = X^2 \supseteq X^1 = S^1 \supseteq X^0 = * \supseteq X^{-1} = \phi$

$\rightsquigarrow$ subspace filtration $\quad Y^2 = S^1 \supseteq Y^1 = S^1 \supseteq Y^0 = * \supseteq Y^{-1} = \phi$

$Y^2 = S^1$ with $\begin{cases} \text{formal dim} & 2 \\ \text{topo dim} & 1 \end{cases}$

不自然的filtration是否因为subspace filtration本身不合理?

We believe it's a natural way to give subspace filtration for Y.

with subspace filtration, $I^{\bar{p}_Y}S_*^{GM}(Y)\subseteq I^{\bar{p}_X}S_*^{GM}(X)$ is a subcomplex $\rightsquigarrow$ we can define
$I^{\bar{p}}S_*^{GM}(X)/I^{\bar{p}_X}S_*^{GM}(Y)=:I^{\bar{p}}S_*^{GM}(X,Y)$

Leading to relative intersection homology □

Filter space is a weak condition, with additional conditions we obtain stratified space (for simplicity we do not use stratified space today)

[Def] The filtered set X satisfies frontier condition if any two strata S,T of X with $S\cap\bar{T}=\varnothing$, then $S\subset\bar{T}$. □

[Exp] Space not satisfies frontier condition

H: upper plane, Y: y-axis. Let $X=H\cup Y$. $X^2=X\supseteq X^1=Y\supseteq\varnothing$

$\overline{X^2-X^1}=H$, $X^1-X^0=Y$. $H\cap Y\neq\varnothing$ but $Y\subsetneq X$.

□

[Rmk] 这个条件本质上在说"the closure of any stratum is a union of strata." □

[Def] (Stratified space) A filtered space satisfying Frontier condition is a stratified space. □

[Rmk] Intersection homology 定义并不要求有 frontier condition, 即不要求 space 是 stratified space. □

- General position. X: simplicial complex.

  i-simplex σ in general position of stratum S if
  $$\dim(\sigma\cap S)\leq\dim(\sigma)+\dim(S)-n$$

  $|\text{simplicial complex}|\simeq$ manifold $\Rightarrow$ It's possible to move σ to be in general position with S

[Exp]

$X^2=T^2\supseteq X^1=*\supseteq X^0=*\supseteq X^{-1}=\varnothing$

strata: $S_1=T^2-*$
$S_2=*$

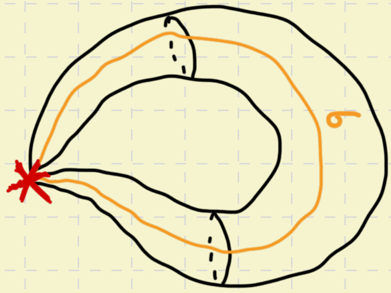
σ is an 1-simplex in picture.

σ in general position with $S_2=\{*\}$ $\Leftrightarrow$ we have $\dim(\sigma\cap S_2)\leq\dim(\sigma)+\dim(S_2)-n$
$=1+0-2=-1$

$\Leftrightarrow *\notin\sigma$

we can always move σ not containing $\{*\}$.

But for pinched torus, it's impossible to move σ not containing $*$!



□

在上个例子中，pinched torus 里的 σ 探测到了空间中的奇点 $*$.
在多大程度上容忍这种怪异的 simplex 可以反映 singular space 的信息.
我们在条件 $\dim(\sigma\cap S)\le\dim(\sigma)+\dim(S)-n$ 的右手边加 调节项 $\bar{p}(S)$ 来刻画容忍程度.

[Def] (Perversity) X: filtered sp of formal dim n
$\mathcal{F}=\{$ strata of $X\}$. A perversity on X is a function
$\bar{p}:\mathcal{F}\to\mathbb{Z}$ s.t. $\bar{p}(S)=0$ if $S\subset X-\Sigma_X$,
i.e., if S is a regular stratum □

[Rmk] Why $\bar{p}(S)=0$ is clear when we consider definition of $\bar{p}$-allowable simplexes. □

[Def] X: simplicial filtered sp with general perversity $\bar{p}$
$C_*(X)$: chain complex of X
i-simplex σ is called $\bar{p}$-allowable if

$$\underbrace{\dim(\sigma\cap S)}_{\text{topo dim}}\le i-\underbrace{\operatorname{codim}(S)}_{\text{formal dim.}}+\bar{p}(S),\ \forall \text{ stratum } S \text{ of } X$$

□

□

[Def] A chain $\xi\in C_i(X)$ is <u>$\bar{p}$-allowable</u> if $\forall$ simplices of ξ and $\partial\xi$ are $\bar{p}$-allowable.

$I^{\bar{p}}C_*(X)=\{\xi\in C_i(X)\mid\xi$ is $\bar{p}$-allowable$\}$

[Rmk] $\xi\in I^{\bar{p}}C_i(X)$, $\forall$ simplex in $\partial\xi$ and $\partial^2\xi=0$ are $\bar{p}$-allowable.
So $\partial\xi\in I^{\bar{p}}C_i(X)$. So $(C_*(X),\partial)$ restricts to chain complex $(I^{\bar{p}}C_*(X),\partial)$

[Def] $I^{\bar{p}}H_*^{GM}(X):=H_*(I^{\bar{p}}C_*^{GM}(X))$

[Rmk] We come back to the question: why $\bar{p}(S)=0$ for $S\subseteq X^n-X^{n-2}$?

① If $\bar{p}(S)<0$ 即 $\bar{p}(S)\le -1$. 令 σ 是一个 i-simplex.

$\dim(\sigma\cap S)\le i+n-n+\bar{p}(S)\le i-1 \Leftrightarrow$ Interior of σ is not contained in any regular stratum S (Interior of σ has dim i)

(Skeleton是闭集) $\Rightarrow \sigma\subseteq X^{n-1} \Rightarrow I^{\bar{p}}H_*^{GM}(X)=I^{\bar{p}}H_*^{GM}(X-S)$

② 若 $\bar{p}(S)\ge 0$

$\dim(\sigma\cap S)\le i+n-n+\bar{p}(S)\le i+\bar{p}(S)$ always holds.

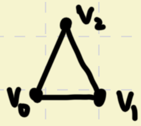
For simplicity, $\bar{p}(S)=0$.

[Rmk] 我们目前定义 GM intersection homology，之后定义 non GM intersection homology. 它们二者的区别在于，non GM intersection homology 的 chain complex $(I^{\bar{p}}C_*(X),\partial_*)$ 会进行一些修正，使得 non GM intersection homology 看起更加正确. □

[Rmk] 有一种条件更强的 perversity 称为 GM perversity，这里不做介绍.
注意，GM intersection homology 不一定拿 GM perversity 来做定义，GM intersection homology 可以使用我们定义的一般的 perversity. GM intersection homology 与 non GM intersection homology 区别见上一个 Rmk.

下面是若干 simplicial intersection homology 的例子.

[Exp1] perversity 对于 intersection homology 的调控未必是敏感的.
当 $\bar{p}(v_0)$ 变化的时候，intersection homology 只有三类结果.

$X=$ 

is the boundary of a 2-simlex , $X=X^1\supseteq X^0=\{v_0\}\supseteq X^{-1}=\phi$.

For $\forall$ 0-simplex v

$$\dim(v\cap\{v_0\})\le \dim v+\dim v_0-n+\bar{p}(v_0) = 0+0-1+\bar{p}(v_0)=\bar{p}(v_0)-1$$

$\bar{p}$-allowable 0-simplex $\begin{cases} v_0,v_1,v_2 & \bar{p}(v_0)\ge 1 \\ v_1,v_2 & \bar{p}(v_0)<1\end{cases}$

For $\forall$ 1-simplex e

$$\dim(e\cap v_0)\le \dim e+\dim v_0-n+\bar{p}(v_0)=1+0-1+\bar{p}(v_0)=\bar{p}(v_0)$$

$\bar{p}$ allowable 1-simplex $\begin{cases} [v_1, v_2], [v_0, v_1], [v_0, v_2] & \bar{p}(v_0) \geq 0 \\ [v_1, v_2] & \bar{p}(v_0) < 0 \end{cases}$

(i) $\bar{p}(v_0) \geq 1$ $\quad I^{\bar{p}}H_*(X) = H_*(X)$

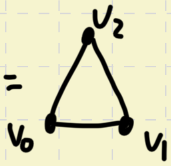
(ii) $\bar{p}(v_0) < 0$ $\quad I^{\bar{p}}H_*(X) = H_*($  $)$

(iii) $\bar{p}(v_0) = 0$ $\quad I^{\bar{p}}H_0(X) = \mathbb{Z}$

$I^{\bar{p}}H_1(X) = \mathbb{Z}$ $\quad (I^{\bar{p}}C_*(X) \subseteq C_*(X))$

cycles in $I^{\bar{p}}C_*(X)$ comes from $C_*(X)$

[Exp] Filtration impacts intersection homology

$X =$ 

$X = X^1 \supseteq X^0 = \{v_0, v_1, v_2\} \supseteq X^{-1}$

singular stratum : v_0, v_1, v_2 .

$\bar{p}(v_i) = 0$

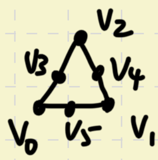
$\forall$ 0-simplex v, $\dim(v \cap v_i) \leq 0 + 0 - 1 + \underbrace{\bar{p}(v_i)}_{=0} = -1$

$\Rightarrow$ all 0-simplex not allowable

$\forall$ 1-simplex e, $\dim(e \cap v_i) \leq 1 + 0 - 1 + \bar{p}(v_i) = 0$ always holds

So $I^{\bar{p}}H_0(X) = 0$, $I^{\bar{p}}H_1(X) = \mathbb{Z}$

[Exp] Subdivision impacts intersection homology

$X =$ 

$X = X^1 \supseteq X^0 = \{v_0, v_1, v_2\} \supseteq X^{-1} = \emptyset$

$\bar{p}(v_i) = 0$

$\dim(v \cap v_i) \leq -1 \rightsquigarrow$ allowable 0-simplexes: v_3, v_4, v_5

$\dim(e \cap v_i) \leq 0 \rightsquigarrow$ all 1-simplexes are allowable.

$I^{\bar{p}}H_1(X) = \mathbb{Z}$, $I^{\bar{p}}H_0(X) = \mathbb{Z}$

[Rmk] 个人理解：perversity 和同调 degree n 很相像，需要计算越多越好，并不是某一个 perversity 是最好的 perversity，只计算那一个 perversity. 事实上，可以定义 dual perversity：

> More generally, if X is any R-oriented locally $(\bar{p}; R)$-torsion-free n-dimensional stratified pseudomanifold, we have a Poincaré duality isomorphism
> $$\mathcal{D} : I_{\bar{p}}H_c^i(X; R) \to I^{D\bar{p}}H_{n-i}(X; R),$$

不仅 degree 上有对偶，perversity 上也有对偶. □

PL Intersection homology

我们先定义 PL homology，再定义 PL intersection homology.

Recall simplicial complex by example:

[Exp] $K_1 = \triangle$, $K_2 = \triangle$ (are simplicial complex.)

$|K_1| = |K_2| = \triangle$ is topo space □

[Def] The simplicial complex K' is a <u>subdivision</u> of K if (i) $|K| = |K'|$
(ii) $\forall$ simplex of $K' \subseteq$ in some simplex of K.

(Exp)

[Def] (Triangulation) A <u>triangulation</u> T of a topo sp X is a pair $T = (K, h)$

K: locally finite simplicial complex

$h : |K| \to X$ be a homeomorphism.

* locally finite: $\forall x \in |K|$, $\exists$ n.b.h. U intersects finite number of simplexes

[Def] Let $T = (K, h)$, $S = (L, j)$ be two triangulations.

$T = (K, h) \sim S = (L, j) \iff j^{-1}h$ is simplicial iso

[Def] (PL space) A PL (piecewise linear) space is a topo sp X with $\mathcal{T} = \{$ locally finite triangulations $\}$ s.t.

(i) $\forall T \in \mathcal{T}$, subdivision of T contained in $\mathcal{T}$

(ii) $\forall T, S \in \mathcal{T}$, T, S has <u>common refinement.</u>

* $T = (K, k)$, $S = (L, l)$. $\exists$ subdivison $T' = (K', k)$ of T, $\exists$ subdivision $S' = (L', l)$ of S. s.t. $l^{-1}k : K' \to L'$ iso.

[Def] (Directed set) A directed set is a pair $(I, \leq)$ where $\leq$ satisfying
(i) transitive
(ii) reflexive
(iii) $\forall a, b \in S, \exists c \in I$ with $a \leq c$ and $b \leq c$

[Rmk] The directed set $(I, \leq)$ can be viewed as a category

$$(I,\leq)\begin{cases} \text{object}: & i \in I \\ \text{mor}: & \mathrm{Hom}_I(i,j) = \begin{cases} * & i \leq j \\ \emptyset & o/w \end{cases} \end{cases}$$

[Def] (Directed diagram) $(I, \leq)$: directed set. $\mathcal{A}b$: cat of ab grps.
A functor $F: I \longrightarrow \mathcal{A}b$ is a directed diagram.

[Rmk]
$$\begin{tikzcd} i \arrow[r, "*"] \arrow[dr, "*"'] & j \arrow[d, "*"] \arrow[dl, phantom, "\circlearrowleft"] \\ & k \end{tikzcd} \quad \xrightarrow{\text{map to}} \quad \begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[dr, "f_{ik}"'] & A_j \arrow[d, "f_{jk}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & A_k \end{tikzcd}$$

I

commutative by composition map in I

commutative by functor F

(directed diagram 是一幅很大的交换图, 箭头是由directed set 的 relation $\leq$ 决定的)

[Fact] (Colimit of direct diagram) $\exists\ (L \in \mathcal{A}b, \{f_i : A_i \to L\}_{i \in I})$ satisfying

(i)
$$\begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[dr, "f_i"'] & A_j \arrow[d, "f_j"] \arrow[dl, phantom, "\circlearrowleft"] \\ & L \end{tikzcd}$$

(directed diagram 加上(L, f_i)后仍交换)

(ii)
$$\begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[d, "f_i"'] \arrow[dr] & A_j \arrow[dl] \arrow[d, "f_j'"] \\ L \arrow[r, dashed, "\exists!"'] & L' \end{tikzcd}$$

($\forall (L', \{f_i'\}_{i \in I})$ rendering diagram commutes,
$\exists!\ L \to L'$ rendering diagram commutes.)

(The following we hope to make $\mathcal{T}$ a directed set)

[Construction] $T = (K, k)$, $S = (L, l) \in \mathcal{T}$

$T \leq S \iff S$ equiv. to a subdivision of T

[Fact] $(\mathcal{T}, \leq)$ is a directed set.

[Construction] $F: \mathcal{T} \longrightarrow \mathcal{A}b$

$$\begin{tikzcd} T = (K, k) \arrow[r] \arrow[d, "*"'] & C_*(K) \arrow[d, "?"] \\ S = (L, l) \arrow[r] & C_*(L) \end{tikzcd}$$

[Rmk]

$$\begin{tikzcd} K \arrow[r] \arrow[dr] & K' \arrow[d] \arrow[dl, phantom, "?"] \\ & K'' \end{tikzcd} \qquad \xrightarrow{F} \qquad \begin{tikzcd} C_*(K) \arrow[r, "?"] \arrow[dr, "?"'] & C_*(K') \arrow[d, "?"] \arrow[dl, phantom, "?"] \\ & C_*(K'') \end{tikzcd}$$

$\mathcal{T}$

(定义?使图交换)

[Construction] $T \le S \overset{\text{Def of} \le}{\Longrightarrow}$ $\exists$ subdiv $T' = (K', k)$ and $l'k : K' \to L$ is iso.

$\Longrightarrow$ induces $C_*(K') \cong C_*(L)$

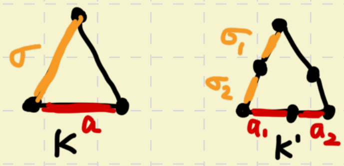
$\Longrightarrow$ We only need to define $C_*(K) \to C_*(K') \cong C_*(L)$

Define $C_*(K) \longrightarrow C_*(K')$

$$\sigma \longmapsto \boxed{\sum_{\tau_\sigma \subseteq |\sigma|} \tau_\sigma}$$

$$\xi = \sum_i a_i \sigma_i \longmapsto \sum_i a_i \sum_{\tau_{\sigma_i} \subseteq |\sigma|} \tau_{\sigma_i}$$

[Exp]



$\sigma \longmapsto \sigma_1 + \sigma_2$

$\sigma + 2a \longmapsto \sigma_1 + \sigma_2 + 2(a_1 + a_2)$

Such map is called subdivision chain map, and it's easy to show

$$\begin{tikzcd} C_*(K) \arrow[r] \arrow[dr] & C_*(K') \arrow[d] \arrow[dl, phantom, "?"] \\ & C_*(K'') \end{tikzcd}$$

Commutes for $K \le K' \le K''$.

[Def] $\mathcal{C}_*(X) = \varinjlim_{T \in \mathcal{T}} C_*^T(X)$, where $C_*^T(X) = C_*(K)$ for $T = (K, k)$

[Rmk] ① $\varinjlim_{T \in \mathcal{T}} C_*^T(X)$ has concrete construction

$$\varinjlim_{T \in \mathcal{T}} C_*^T(X) = \bigcup_{T \in \mathcal{T}} C_*^T(X) \Big/ \sim$$

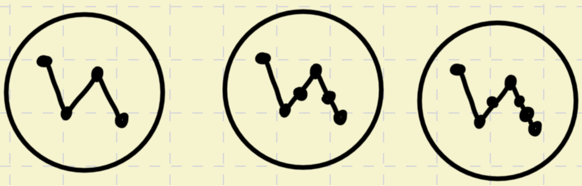
where $\xi \sim \eta \Leftrightarrow \xi$ and η maps to same image in $\varinjlim_{T \in \mathcal{T}} C_*^T(X)$

$$\left(\begin{tikzcd} \xi \; C_*^T(X) \arrow[dr] & & \eta \; C_*^S(X) \arrow[dl] \\ & \varinjlim_{T \in \mathcal{T}} C_*^T(X) & \end{tikzcd} \right)$$

Let $[\xi]$ denote the equiv. class.

② $[\xi] = [\eta]$ iff their image agree in some common subdivision.

e.g.



are same elements in $\mathcal{C}_*(X)$

2. Let I be a directed set, L an abelian group, and $A\colon I \to \mathbf{Ab}$ an I-directed diagram of abelian groups, with bonding maps $f_{ij} : A_i \to A_j$ for $i \leq j$. Show that a map $A \to c_L$, the constant functor at L, given by compatible maps $f_i\colon A_i \to L$, is a direct limit if and only if

(a) for any $b \in L$ there exists $i \in I$ and $a_i \in A_i$ such that $f_i a_i = b$, and

(b) for any $a_i \in A_i$ such that $f_i a_i = 0 \in L$, there exists $j \geq i$ such that $f_{ij} a_i = 0 \in A_j$.

(b) 若 $a,b \in A_i$ map到L的同一个点，则必存在 j s.t. a,b map到 A_j的某一个点

⑤ X is a PL space with admissible triangulations $\mathcal{T}$.

Let $T_0 = (K,h) \in \mathcal{T}$ and let $\mathcal{T}_0 = \{T \in \mathcal{T} \mid T \text{ subdivision of } T_0\}$

Then
$$C_*(X) = \varinjlim_{T\in\mathcal{T}} C_*^T(X) \cong \varinjlim_{T\in\mathcal{T}_0} C_*^T(X)$$

[Def] X : PL space. Define $\mathfrak{S}_*(X) := H_*(C_*(X))$

[prop] X : PL space. $\mathfrak{S}_*(X) \cong H_*(X)$, where $H_*(X)$ can be singular or simplicial homology w.r.t. $\forall$ triangulation.

[Rmk] PL intersection homology $\overset{?}{\neq}$ $H_*(X)$

[Def] X : PL filtered sp s.t. $\forall$ skeleton X^i is a subcomplex of any admissible triangulation.

Define $I^{\bar{p}}C_*(X) = \varinjlim_{T\in\mathcal{T}} I^{\bar{p}}C_*^{GM,T}(X)$, where $I^{\bar{p}}C_*^{GM,T}(X) := I^{\bar{p}}C_*^{GM}(|K|)$

[Rmk] : Skeleton can inherit triangulation from X w.r.t. any admissible triangulation.

[Rmk] filtration & perversity of X can "move to" $|K|$ by homeo k.

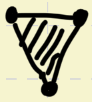
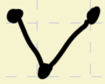
[Fact] $T \leq T'$ subdivision chain map $\nu : C_*^T(X) \to C_*^{T'}(X)$ restricts to a map $\nu : I^{\bar{p}}C_*^{GM,T}(X) \to I^{\bar{p}}C_*^{GM,T'}(X)$

[Def] $I^{\bar{p}}\mathfrak{S}_*^{GM}(X) = H_*(I^{\bar{p}}C_*^{GM}(X)) \cong \varinjlim_{T\in\mathcal{T}} H_*(I^{\bar{p}}C_*^{GM,T}(X))$
$$= \varinjlim_{T\in\mathcal{T}} I^{\bar{p}}H_*^{GM,T}(X)$$

[prop] Let $\xi \in C_i(X)$.

$$\xi \in I^{\bar{p}}C_i(X) \iff \begin{cases} \dim(|\xi| \cap S) \leq i - \operatorname{codim} S + \bar{p}(S) \\ \dim(|\partial\xi| \cap S) \leq i-1-\operatorname{codim} S + \bar{p}(S) \end{cases} \text{ for } \forall \text{ stratum } S \text{ of } X$$

[Def] $L \subseteq K$. L is called full subcomplex if
$\forall \sigma \in K$ with vertices in $L \Rightarrow \sigma \in L$

[Exp] $K =$  , $L =$  L is NOT full subcomplex of K.

[Def] (Full triangulation) An admissible triangulation T of PL filtered space X is called <u>full triangulation</u> if
$\forall X^i$ is full subcomplex of X.

[Thm] X : PL filtered space.
T : full triangulation.
T' : any subdivision of T.
Then $I^{\bar{p}} C_*^{GM,T} \longrightarrow I^{\bar{p}} C_*^{GM,T'}(X)$ is an iso

[Coro]
$$\begin{aligned} I^{\bar{p}} \Omega_*^{GM}(X) &= H_*(I^{\bar{p}} C_*^{GM}(X)) \\ &= H_*(\varinjlim_{T \in \mathcal{T}_0} I^{\bar{p}} C_*^{GM,T}(X)) \\ &\cong H_*(I^{\bar{p}} C_*^{GM,T}(X)) = I^{\bar{p}} H_*^{GM,T}(X) \end{aligned}$$

(No example for computing PL intersection homology. 只要取一个 full triangulation, 就回到 simplicial intersection homology)

Singular homology

[Def] X : filtered space with general perversity $\bar{P}$
$S_*(X)$: singular chain complex of X, i.e., $S_i(X) = \{\Delta^i \to X\}$

A singular i-simplex $\sigma : \Delta^i \to X$ is called $\bar{p}$-allowable if

$\sigma^{-1}(S) \subseteq \{(i - \mathrm{codim}(S) + \bar{p}(S)) - \text{skeleton of } \Delta^i\}$ for all strata S of X.

A chain $\xi \in S_i(X)$ is $\bar{p}$-allowable if all of the simplices in ξ and all of the simplices of $\partial\xi$ are $\bar{p}$-allowable.

Let $I^{\bar{p}} S_*^{GM}(X) = \{\xi \in S_*(X) \mid \xi \text{ is } \bar{p}\text{-allowable}\}$

Define singular intersection homology $I^{\bar{p}} H_*^{GM}(X) = H_*(I^{\bar{p}} S_*^{GM}(X))$

[Exp] (Singular homology 可以和 simplicial homology 相同)

X is a simplicial filtered space, and the singular simplex $\sigma \hookrightarrow X$ is inclusion. 则 $\sigma^{-1}(S) = \sigma \cap S$. 故 $\dim(\sigma^{-1}(S)) \le i - \operatorname{codim}(S) + \bar{p}(S)$

等价于 $\dim(\sigma \cap S) \le i - \operatorname{codim}(S) + \bar{p}(S)$

(Singular homology is not easy to compute by hand, so there is no more appropriate examples)

Theorem 5.4.2 *Let X be a PL filtered space with triangulation T, and let $W \subset X$ be an open subset of X such that W is a PL CS set. Then the composition*

$$I^{\bar{p}}\mathfrak{H}_*^{GM}(W;G) \xrightarrow{\theta^{-1}} I^{\bar{p}}\mathfrak{H}_*^{GM,T}(W;G) \xrightarrow{\psi} H_*(I^{\bar{p}}\mathfrak{S}_*^{GM}(W;G))$$

is an isomorphism. In particular, $I^{\bar{p}}\mathfrak{H}_*^{GM}(W;G) \cong I^{\bar{p}}H_*^{GM}(W;G)$, and if X is a PL CS set then $I^{\bar{p}}\mathfrak{H}_*^{GM}(X;G) \cong I^{\bar{p}}H_*^{GM}(X;G)$.

Big picture

Relationship: $\text{simplicial} \underset{\text{full triangulation}}{=\!=} \text{PL} \underset{\text{"some triangulation"}}{=\!=} \text{Singular}$

(Thm 5.4.2 in Ref)

对于 Intersection homology，在对普通的条件进行修正后，会得到平行的结论.

e.g.

ordinary homology

$f: X \to Y$ is a homotopy equiv, then $f_*: H_*(X) \xrightarrow{\cong} H_*(Y)$

PL intersection homology

$f: X \to Y$ is a stratified homotopy equiv and $\bar{p}_X(S) = \bar{q}_Y(T)$ if $f(S) \subseteq T$.

Then f induces

$$I^{\bar{p}}H_*^{GM}(X) \cong I^{\bar{q}}H_*^{GM}(Y)$$

[Rmk] For more props, see ch 4&5 in Ref.

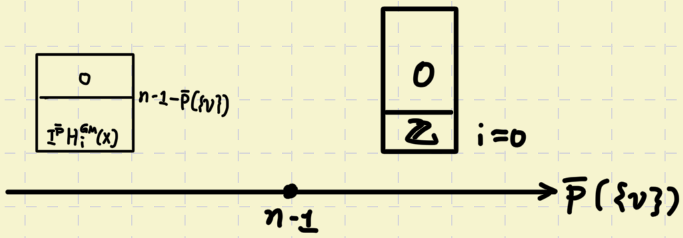
[Rmk] 和 ordinary singular homology 类似, singular intersection homology is not easy to compute by hand. 因此计算 singular intersection homology 时需要使用 tools like "Intersection version of Mayer seq" see Ch 5 in Ref.

[Rmk] 用类似的思路也许可以得到 Intersection homotopy.

Non GM intersection homology

[Exp] X : compact $(n-1)$-dimensional filtered space and assume X has regular strata (so $\exists$ allowable 0-simplex s.t. $I^{\bar{p}}H_0^{GM}(X) \neq 0$)

$$I^{\bar{p}}H_i^{GM}(cX) \cong \begin{cases} 0 & i \ge n - \bar{p}(\{v\}) - 1,\ i \neq 0 \\ \mathbb{Z} & i = 0 \ge n - \bar{p}(\{v\}) - 1 \\ I^{\bar{p}}H_i^{GM}(X) & i < n - \bar{p}(\{v\}) - 1 \end{cases}$$



When you look carefully, you may think it's strange.

当 $\bar{p}(\{v\}) < n-1$ 时, 以 $i = n-1-\bar{p}(\{v\})$ 为界, $i \geq n-1-\bar{p}(\{v\})$ $I^{\bar p}H_i = 0$, $i < n-1-\bar{p}(\{v\}) = I^{\bar p}H_i^{GM}(X)$. 此时随着 $\bar{p}(\{v\})$ 增大, 会越早出现 0.

在极限情形下, 即 $\bar{p}(\{v\}) = n-1$ 时, 应当有从0阶开始所有同调群都是 0.

但事实是0阶同调群是 $\mathbb{Z}$.

It suggests that GM intersection homology done well for "small" $\bar p$, but not right for "large" $\bar p$!

[Exp] This example show you why GM intersection homology is not "right" homology theory.

M: n-dim ∂-mf with $\partial M \neq \varphi$.

$M^+ := M \cup_{\partial M} \bar{c}(\partial M)$ with cone pt v.

$$I^{\bar p}H_i^{GM}(M^+) \cong \begin{cases} H_i(M, \partial M) & i > n-\bar{p}(\{v\})-1 \\ \mathrm{Im}(H_i(M) \to H_i(M,\partial M)) & i = n-\bar{p}(\{v\})-1 \\ H_i(M) & i < n-\bar{p}(\{v\})-1 \end{cases}$$

$$\frac{\text{relative homology grp}}{\text{absolute homology grp}} \quad i = n-\bar{p}(\{v\})-1$$

当 $\bar{p}(\{v\})$ 足够大, 则我们期待在 degree 0 处看到 relative homology behavior 但这不符合事实.

[Idea] 尝试引入 Non GM intersection homology

- Behavior more like relative group
- Agree with GM intersection homology for small perversity.

改造 singular chain complex, we hope it behavior as relative singular chain complex $S_*(X, \Sigma; G)$ with coeffient G. 因此, 落在 Σ 中的 simplex 需要扔掉, 因为它在 $S_*(X, \Sigma; G)$ 中是 0.

[Def] Let $S_i^{\bar{p}}(X;G) \subseteq S_i(X;G)$ generated by $\bar{p}$-allowable i-simplex σ with support $|\sigma| \not\subset |\Sigma|$.

[Def] $\hat{\partial}\sigma = \sum_{|\sigma_j| \not\subset \Sigma_X} (-1)^j \sigma_j$

[Rmk] $\hat{\partial}\sigma$ is obtained from $\partial\sigma$ by throwing out the simplices with image in Σ.

[Def] Let $I^{\bar{p}}S_i(X;G) = \{ \xi \in S_i^{\bar{p}}(X;G) \mid \hat{\partial}\xi \in S_{i-1}^{\bar{p}}(X;G) \}$.

$(I^{\bar{p}}S_i(X;G), \hat{\partial})$ is a chain complex, and then we define non-GM intersection homology $I^{\bar{p}}H_*(X;G) = H_*(I^{\bar{p}}S_*(X;G))$.

[Rmk] 对 simplicial 与 PL intersection homology 有类似的定义.