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Around Algebraic Cycles

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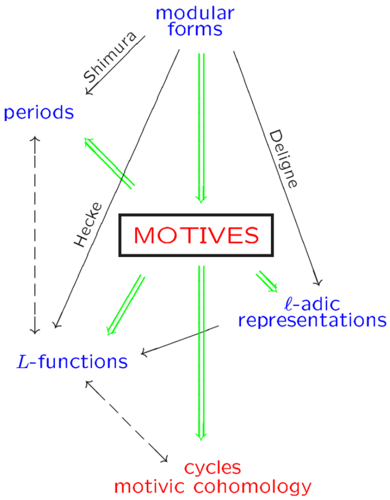
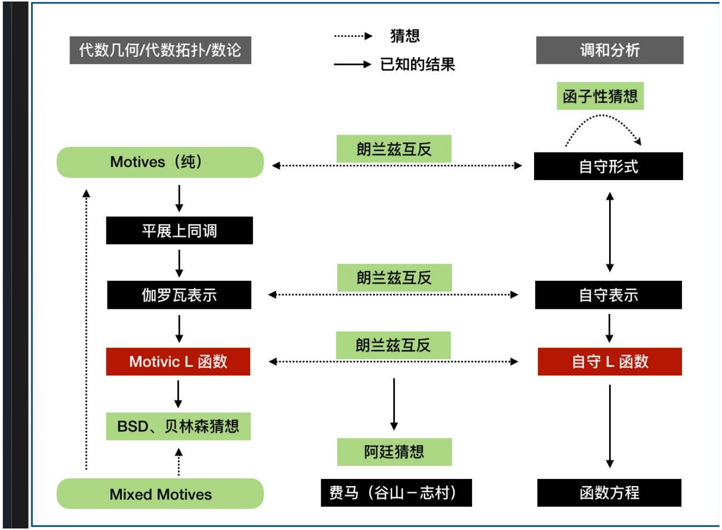
Introduction





Cycles

In this section, schemes are confined to smooth projective. We fix a d-dimensional scheme X.

Definition

An algebraic cycle on a scheme X is a formal integral linear combination

$$Z = \sum n_i Z_i$$

of closed integral subschemes Z_i.

If all Z_i have the same codimension d, we say that Z is a codimension d cycle. We denote by $\mathcal{Z}^d(X)$ the free abelian group of codimension d cycles on X.

Here are some simple examples:

Example

The group of codimension 1-cycles is also called Weil divisors.

The zero cycles $\mathcal{Z}^d(X)$. There are just formal sums of closed points $Z = \sum_i n_i P_i$ where P_i is a closed point. The degree of P_i is just the degree of the field extension $[k(P_i) : k]$ and $\deg(Z) = \sum_i n_i \deg(P_i)$.

Let Y be a subscheme of X with irreducible components Y_i of dimension d_i. We can define the cycle

$$[Y] = \sum_i n_i Y_i$$

where n_i is the length of the zero dimension Artinian ring $\mathcal{O}_{Y,Y_i}$.

Let $Y = V(x^2 y)$ be a closed subscheme of $\mathbb{A}_k^2$. We have $[Y] = 2[V(x)] + [V(y)]$.
(For calculation of length, we prefer the free computation program "macaulay2")

Definition (Intersection)

Let Z and T be two subschemes of X. The intersection is $Z \cap T := \Delta^*(Z \times_k T)$.

Theorem (Serre)

For any closed integral subscheme Z and T of X, we have

$$codim_X(Z) + codim_X(T) \geq codim_X(Z \cap T)$$

When equality occurs, we say Z and T intersect properly. Otherwise, excess.

Definition (correspondences)

A correspondence from X to Y is a cycle in $X \times_k Y$.

A correspondence $Z \in \mathcal{Z}^t(X \times_k Y)$ acts on cycles $T \in \mathcal{Z}^i(X)$ as follows:

$$Z(T) := [pr_Y]_*(Z \cap (T \times_k Y)) \in \mathcal{Z}^{i+t-d}(Y)$$

Galois invariance of cycles

Given a field extension $k \subset L$. There is a base change homomorphism $\mathcal{Z}^d(X) \to \mathcal{Z}^d(X \times_k L)$.

Proposition

The homomorphism $\mathcal{Z}^d(X) \to \mathcal{Z}^d(X \times_k L)$ is injective.

If L is Galois over K. Then $\mathcal{Z}^d(X) \xrightarrow{\sim} \mathcal{Z}^d(X \times_k L)^{Gal(L/k)}$.

Proof.

(1) Let Z_1 and Z_2 be different codimension d integral closed subschemes of X. Then the schemes $Z_i \times_k L$ are nonempty and do not intersect. Otherwise, $Z_1 \times_k L \cap Z_2 \times_k L$ descents to a subscheme of X. (2) Consider a Gal(L/k)-orbit of $\mathcal{Z}^d(X)$. Let Y be their union. It is a closed subscheme of X_L. It descends to a subscheme of X. Since classes $[Y]$ form a basis of $\mathcal{Z}^d(X_L)^{Gal(L/k)}$, it is an isomorphism. ∎

Cartier divisors

Let X be a scheme. If $U \subset X$ is an open sub set, a section $f \in \mathcal{O}_X(U)$ is said to be regular if for every $x \in U$, its image in the stalk $\mathcal{O}_{X,x}$ is a non-zero divisor. The regular sections form a subsheaf $\mathcal{O}_{X,reg}$ of the sheaf of monoids (with respect to multiplication) $\mathcal{O}_X$. The sheaf of total quotient rings κ_X is the localization of $\mathcal{O}_X$ with respect to $\mathcal{O}_{X,reg}$.

Definition

Write Div_X for the sheaf $\kappa_X^*/\mathcal{O}_X^*$. The group $H^0(X, Div_X) = Div(X)$ is the Cartier divisor group.

Proposition (EGA 4, section 21.6)

There exists a unique homomorphism of sheaves: div $: Div_X \to \mathcal{Z}_X^1$ *such that, if* $U \subset X$ *is an open set and* $f \in \mathcal{O}_{X,reg}(U)$*, then* $div(f) = [\mathcal{O}_U/(f)]$*, the cycle associated to the codimension one subscheme. If* X *is regular, div is an isomorphism.*

To get some interesting and useful properties on cycles. We import equivalence relations between them.

Definition (rational equivalence)

The divisor map *div* maps $\bigoplus_{x \in X^{(p-1)}} \kappa(x)^*$ to $\bigoplus_{x \in X^{(p)}} \mathbb{Z}$. We say that a cycle in the image of *div* is rationally equivalent to zero. The Chow group of X is the cokernel of *div*.

Definition (equivalent definition)

A cycle $Z \in \mathcal{Z}^i(X)$ is rationally equivalent to zero if there is a cycle $W \in \mathcal{Z}^i(X \times_k \mathbb{P}^1_k)$ and $a, b \in \mathbb{P}^1_k$ such that $W(a) = 0$ and $W(b) = Z$. We denote by $\mathcal{Z}^i_{rat}(X)$ the set of all cycles that are rationally equivalent to zero.

If we replace $\mathbb{P}^1_k$ by a smooth irreducible curve, we get the notion of **algebraic equivalence.** Every rational equivalent cycle is also an algebraic equivalent. Any two distinct points on an elliptic curve are algebraic equivalent but not rational equivalent.

Basic properties of Chow groups

Proposition

Let $i : Y \to Z$ be a closed subscheme of X, and $U = X - Y$ the open complementary. Then the following sequence is exact:

$$CH^q(Y) \to CH^q(X) \to CH^q(U) \to 0.$$

For a proper morphism $f : X \to Y$, we have additive homomorphisms

$$f_* : CH^q(X) \to CH^q(Y)$$

Let $f : X \to Y$ be a flat of relative dimension n. We have homomorphisms

$$f^* : CH_q(Y) \to CH_{q+n}(X)$$

$CH^p(X \times \mathbb{A}^1_k) = CH^p(X)$.

Some calculations

Example

$CH_i(\mathbb{A}_k^n) = 0$ for all $i < n$. But $CH_n(\mathbb{A}_k^n) = \mathbb{Z}$. $CH_n(\mathbb{P}_k^n) = \mathbb{Z}$ and $CH_{n-1}(\mathbb{P}_k^n) = \mathbb{Z}$.

Let X be the compact toric surface corresponding to the fan in $\mathbb{Z}^2$ with edges through the points $(2, -1), (-1, 2), (-1, -1)$. Then $CH_1(X) = \mathbb{Z} \oplus \mathbb{Z}/3$.

Consider the fan with edges through the vertices $(\pm 1, \pm 1, \pm 1)$ of a cube in the subspace of $\mathbb{Z}^3$ generated by theses vertices. Then the corresponding compact toric threefold X has

$$CH_0(X) = \mathbb{Z}, CH_3(X) = \mathbb{Z}, CH_1(X) = \mathbb{Z}, CH_2(X) = \mathbb{Z}^5.$$

Generally, it is not true that $CH^i(X) = CH^i(X \times_k k')^{Gal(k'/k)}$.
Let k_s be a separable closure of k. We have the Hochschild-Serre spectral sequence:

$$H^p_{Gal}(Gal(k_s/k); H^q_{\text{ét}}(X \times_k k_s, \mathcal{F})) \Rightarrow H^{p+q}_{\text{ét}}(X; \mathcal{F}).$$

Taking $\mathcal{F} = \mathbb{G}_m$, we have the following exace sequence:

$$0 \to \text{Pic}(X) \to \text{Pic}(X \times_k k_s)^{Gal(k_s/k)} \to \text{Br}(k) \to \ker(\text{Br}(X) \to \text{Br}(X \times_k k_s)).$$

Example

Let $k = \mathbb{R}, k_s = \mathbb{C}$, and X the curve given by $X^2 + Y^2 + Z^2 = 0$. Then X is isomorphic to $\mathbb{P}^1$ over k_s but not over k. Therefore, $\text{Pic}(X \times_k k_s) = \mathbb{Z}$. Since $Gal(k_s/k) = \mathbb{Z}/2$, $\text{Pic}(X)$ has index at most 2 inside $\text{Pic}(X \times_k k_s)$. The line bundle $\mathcal{O}(1)$ on $X \times_k k_s$ does not descend to X. Therefore $Gal(k_s/k)$ acts trivially on $\text{Pic}(X \times_k k_s)$ given the exact sequence

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Néron-Severi group

Suppose X is a projective scheme over $\mathbb{C}$ or a compact complex manifold. Recall the exponential sequence

$$0 \to \mathbb{Z}(1) := 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

The associates long exact sequence is

$$H^1(X, \mathbb{Z}(1)) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1)) \xrightarrow{j} H^2(X, \mathcal{O}_X)$$

Define $\mathrm{Pic}^0(X) := \ker c_1$. The quotient $NS(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is the **Néron-Severi** group of X. One can show that $NS(X) \cong \ker(j)$, $H^2(X, \mathbb{Z}(1))$ is finitely generated, hence so is $NS(X)$.
We also have similar facts on general algebraically closed fields. But it need some delicate theory.

Definition

Let X be a proper scheme over an algebraically closed field k. The **Néron-Severi** group of X is the abelian group

$$NS(X) := \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^0_{X/k}(k)$$

Definition

Néron-Severi scheme of a proper scheme X over K is the commutative group scheme

$$NS_{X/k} := \mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k,red}$$

When k is algebraically closed, we recover the Néron-Severi group as the group of k-points

$$NS_{X/k}(k) = \mathrm{Pic}_{X/k}(k)/\mathrm{Pic}^0_{X/k}(k) = NS(X).$$

Theorem of the Base

Theorem (SGA 6, Exposé XIII)

Let X be a smooth projective scheme over an algebraically closed field k. Then $NS(X)$ is a finitely generated abelian group.

As a corollary

Theorem

Let X be a proper scheme over an algebraically closed field K. Then $NS(X)$ is a finitely generated abelian group.

The Albanese scheme $\mathrm{Alb}_X := (\mathrm{Pic}^0_{X/k})^\vee_{red}$ is the universal abelian scheme. One has $H^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \cong \mathrm{Alb}_X(k)_{\ell-tor}$ for a d-dimensional smooth scheme over a separably closed field. See Shinichi Mochizuki's paper "TOPICS IN ABSOLUTE ANABELIAN GEOMETRY I" for a scheme theoretic account.

An explanation of $NS(X)$

A Riemann form for a complex torus X is a Hermitian form $H : V \times V \to \mathbb{C}$ for which the alternating form $E = \mathrm{Im}H : V \times V \to \mathbb{R}$ is integer-valued on $\Gamma \times \Gamma$.

Theorem

For a complex torus $X = V/\Gamma$, $NS(X)$ is Riemann forms on X.

sketch.

$$0 \to NS(X) \to H^2(X, \mathbb{Z}(1)) \xrightarrow{f} H^2(X, \mathcal{O}_X)$$

Note that $H^2(X, \mathbb{Z})$ =alternating bilinear $E : \Gamma \times \Gamma \to \mathbb{Z}$, and $H^2(X, \mathcal{O}_X) = Alt^2_{\mathbb{C}}(\overline{V})$. Under the isomorphism $H^2(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}(1))$ and $H^2(X, \mathcal{O}_X) \xrightarrow{2\pi i} H^2(X, \mathcal{O}_X)$, f takes $E \in Alt^2_{\mathbb{Z}}(\Gamma)$ to $E_{\mathbb{C}}|_{\overline{V}\times\overline{V}} \in Alt^2_{\mathbb{C}}(\overline{V})$. Hence

$$NS(X) \simeq \{E \in Alt^2_{\mathbb{Z}}(\Gamma) | E_{\mathbb{C}}|_{\overline{V}\times\overline{V}} = 0\} = \{Riemann\ forms\}$$

Studying Chow groups by bare definition is hard. There are some cohomological methods. Let $\mathcal{K}_i$ be the Zariski sheaf associate to the presheaf $U \mapsto K_i(U)$. We have the following fourth quadrant spectral sequence

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{K}_{-q}) \Rightarrow K_{-p-q}(X).$$

We have $H^p_{Zar}(X, \mathcal{K}_p) \cong CH^p(X)$ for all $p > 0$ (Bloch-Quillen formula). This comes from the Gersten resolution of $\mathcal{K}_n$:

$$0 \to \mathcal{K}_n \to \bigoplus_{cd=0} (i_x)_* K_n(x) \to \bigoplus_{cd=1} (i_y)_* K_n(y) \to \cdots \to \bigoplus_{cd=n-1} (i_z) K_n(z) \to \bigoplus_{cd=n} (i_w) K_n(w) \to 0.$$

Similarly, let $\mathcal{H}^i(\mu_n^{\otimes j})$ be the Zariski sheaf associate to the presheaf $U \mapsto H^i_{\text{ét}}(U, \mu_n^{\otimes j})$. We get the spectral sequence

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{H}^q(\mu_n^{\otimes j})) \Rightarrow H^{p+q}_{\text{ét}}(X, \mu_n^{\otimes j}), E_2^{p,p} = CH^p(X)/n.$$

The term $E_2^{p,q}$ concentrate on $0 \leq p \leq \dim(X)$ and $p \leq q$.

Define the Milnor K-theory sheaf:

$$\mathcal{K}^M_{r,X} := (\mathcal{O}^*_X \otimes \cdots \otimes \mathcal{O}^*_X)/J.$$

where J is the subsheaf of the tensor product generated by sections of the form

$$\{\tau_i \otimes \cdots \otimes \tau_r | \tau_i + \tau_j = 1, j \neq i\}.$$

Theorem (Soulé, 1985)

For any field k, we have $H^p_{Zar}(X, \mathcal{K}^M_p) \otimes \mathbb{Q} \cong CH^p(X) \otimes \mathbb{Q}$.

Theorem (Kerz's thesis, 2009)

If k is an infinite field, we actually have $H^p_{Zar}(X, \mathcal{K}^M_p) \cong CH^p(X)$.

More equivalence relations

Definition (Homological equivalence)

Let H be a Weil cohomology (examples include ℓ-adic cohomology and singular cohomology). One has a functorial cycle class map

$$\gamma_X : CH^i(X) \to H^{2i}(X).$$

A cycle $Z \in \mathcal{Z}^i(X)$ is **homologically equivalent** to zero if $\gamma_X(Z) = 0$. We denote by $\mathcal{Z}^i_{hom}(X)$ all homologically equivalent to zero cycles.

The cycle map compatible with intersection product: $\gamma_X(\alpha \cap \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$. Consequently:

Proposition

We have $\mathcal{Z}^i_{alg}(X) \subset \mathcal{Z}^i_{hom}(X)$.

In general, algebraical equivalence and homological equivalence are different. Griffiths gave the first example.

Theorem (Griffiths(1969), Clemens(1983))

Let $X = V_+(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5) \subset \mathbb{P}^5_{\mathbb{C}}$*. Consider these 3 copies of* $\mathbb{P}^2_{\mathbb{C}} \subset X$*:* $L_1 = V_+(z_0 + z_1, z_2 + z_3, z_4 + z_5)$*,* $L_2 = V_+(z_0 + \xi z_2, z_2 + \xi z_3, z_4 + z_5)$*, and* $L_3 = V_+(z_0 + \xi z_1, z_2 + \xi z_3, z_4 + \xi z_5)$ *where* ξ *is a primitive* 5*-th root of unity. Then* $L_3 \cap (L_1 - L_2) = 1 \neq 0$*. Hence* $L_1 - L_2$ *is not homologically equivalent to zero.*
Further, Clemens showed that if

$$Y = V_+(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + (\sum_{j=0}^{4} a_j z_j)^5) \subset \mathbb{P}^4_{\mathbb{C}},$$

for $a_0, a_1, a_2, a_3, a_4 \in \mathbb{C}$*, then* $\mathcal{Z}^2_{hom}(Y)/\mathcal{Z}^2_{alg}(Y) \neq 0$ *contains an infinite cyclic subgroup.*

More equivalence relations

Definition (Numerical equivalence)

Let X be a d dimensional scheme. For $Z \in \mathcal{Z}(X)^i$, we say Z is numerically equivalent to zero if for every $W \in \mathcal{Z}^{d-i}(X)$ such that $Z \cap W$ is defined, we have $\deg(Z \cap W) = 0$. Let $\mathcal{Z}_{num}(X) \subset \mathcal{Z}^i(X)$ be the subgroup of the cycles numerically equivalent to 0.

Theorem (Matsuaka)

Homological equivalence and numerical equivalence coincide on codimension one cycle.

Theorem (Lieberman(1968))

Over $\mathbb{C}$. Homological and numerical equivalence coincide, besides in codimension 1, in codimension 2, and in dimension 1. On abelian varieties homological equivalence and numerical equivalence coincide.

Exercise: Using Bertini theorem, demonstrate that numerical equivalence and algebraic equivalence coincide for zero cycles.

Theorem

If k is algebraically closed. Then $\mathcal{Z}^i(X)/\mathcal{Z}^i_{num}(X)$ is finite dimensional. (it is automatically torsion free)

Proof.

Since $H^{2d-2i}_{\text{ét}}(X, \mathbb{Q}_\ell(d-i))$ is finite, we can choose $a_1, \ldots, a_m \in \mathcal{Z}^{d-i}(X)$ whose classes under γ_X form a maximal set of $\mathbb{Q}_\ell$-linearly independent elements. Consider the linear map

$$\lambda : \mathcal{Z}^i(X) \to \mathbb{Z}^m, \beta \mapsto (\deg(\beta \cap a_1), \ldots, \deg(\beta \cap a_m)).$$

One can show $\ker(\lambda) = \mathcal{Z}^i_{num}(X)$. Hence λ induces an injection

$$\mathcal{Z}^i(X)/\mathcal{Z}^i_{num}(X) \otimes \mathbb{Q} \to \mathbb{Q}^m.$$

Let X be a smooth, geometrically connected, projective, dimension n scheme over $\mathbb{F}_q$. We put

$$Z(X,t) = \prod_{x \text{ closed}} (1 - t^{\deg(x)})^{-1}$$

According to Grothendieck, Artin, Verdier, and Deligne

$$Z(X,t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)}$$

with $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$, and for $1 \leq i \leq 2n-1$, we have $(X,t) \in \mathbb{Q}[t]$ and $P_t(t) \in \mathbb{Z}[t]$,

$$P_i(t) = \prod_{j=1}^{\dim H^i(X,\mathbb{Q}_\ell)} (1 - \alpha_{ij} t)$$

with α_{ij} algebraic integer with $|\alpha_{ij}| = q^{i/2}$.

John Tate made the following conjecture

Conjecture

The order of poles of $Z(X, t)$ at d is equal to the rank of $\mathcal{Z}^d(X)/\mathcal{Z}^d_{num}(X)$.

Conjecture (Beilinson)

Keep the above assumptions, homological equivalence and numerical equivalence agree on $\mathcal{Z}^d(X) \otimes \mathbb{Q}$.

It is easy to show that homological equivalence is numerical equivalence. But the conjectural inverse or any possible counterexample is highly non-trivial. In fact, there is an old conjecture:

Conjecture (Grothendieck's standard conjecture $D(X)$)

Suppose k is algebraically closed. Then

$$\mathcal{Z}^i_{hom}(X) = \mathcal{Z}^i_{num}(X).$$

This conjecture is extremely hard. It is one of the major problems in algebraic geometry to achieve some understanding of the subspace generated by the cycle classes. We have two major conjectures, namely the Hodge and the Tate conjecture which once they have been proved yield a description of the image.

Motivation for Motives

In algebraic geometry, there are many cohomology theories, ℓ-adic ones, one for each ℓ, cristalline cohomology, de Rham cohomology, etc. Grothendieck envisioned universal coefficients which would dictate every other cohomology theory for schemes. Philosophically, motives are objects in a **universal abelian category** $Mot(k)$ attached to the category of algebraic varieties whose most important property is to have cohomology: singular and de Rham cohomology for axamples. Every variety has a motive $h(X)$ which should decompose into components $h^i(X)$ for $i = 0, \ldots, 2\dim x$. Singular cohomology of $h^i(X)$ is concentrated in degree i and equal to $H^i_{sing}(X^{an}, \mathbb{Q})$ for example. All standard properties of cohomology are assumed to be induced by properties of the category of motives: the Künneth formula for the product of two varieties is induced by a tensor structure on motives; Poincare duality is induced by the existence of strong duals on motives. Impressive progress has been made. For **pure motives**, the motives of smooth projective schemes, Grothendieck gave an unconditional construction. But some expected properties depend on the standard conjecture. Similar obstacle occurs in the hoped for motivic t-structure on **Voevodsky's triangulated category of mixed motives**.

A words of Grothendieck

Grothendieck: In order to express this intuition, of the kinship of these different cohomological theories, I formulated the notion of “motive” associated to an algebraic variety. By this term, I want to suggest that it is the “common motive” (or “common reason”) behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible.

Grothendieck's construction

Grothendieck's construction is based on the notion of correspondence.

Definition

Fix an equivalence relation $\sim$. Let X_d and Y_e be two irreducible smooth projective schemes. The group of correspondences between X and Y is

$$Cor_{\sim}(X, Y) := CH_{\sim}(X \times_k Y; \mathbb{Q}).$$

One defines the degree of correspondences by

$$Cor^r_{\sim}(X, Y) := CH^{d+r}_{\sim}(X_d \times_k Y; \mathbb{Q}).$$

Let $f \in Cor_{\sim}(X, Y)$ and $g \in Cor_{\sim}(Y, Z)$ the $g \circ f \in Cor_{\sim}(X, Z)$ is defined by

$$g \circ f := Pr_{XZ}((f \times Z) \cap (X \times g)).$$

Definition

A correspondence $p \in Cor^0_\sim(X, X)$ is a projector of X if $p \circ p = p$.

Definition (Effective pure motives)

This is a category $\mathcal{M}^+_\sim(k)$ which has objects pairs (X, p) with $X \in SmProj_k$ and p a projector of X and morphisms if $M = (X, p)$, $N = (Y, q)$,

$$\mathrm{Hom}(M, N) := q \circ Cor^0_\sim(X, Y) \circ p, X \xrightarrow{p} X \xrightarrow{f} Y \xrightarrow{q} Y$$

Definition (Pure motives)

The category $\mathcal{M}_\sim(k)$ has objects are triples $M = (X, p, m)$ with $X \in SmProj_k$, p a projector of X and $m \in \mathbb{Z}$, and with morphisms, if $M = (X, p, m)$, $N = (Y, q, n)$,

$$\mathrm{Hom}(M, N) := q \circ Cor^{n-m}_\sim(X, Y) \circ p$$

Takes $\sim$ the rational equivalence. We get the so called **Chow motives** $CH\mathcal{M}(k)$.
Fix a Weil cohomology (say $H_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ or $H_{Betti}(X(\mathbb{C}), \mathbb{Q})$). Take $\sim$ for the corresponding homological equivalence. We get the homological motives.
Finally, take numerical equivalence we get the **Grothendieck motives** $\mathcal{M}_{num}(k)$.

Definition (Cycle groups for motives)

Let $M = (X, p, m) \in \mathcal{M}_\sim(k)$. The correspondence p operates on the vector spaces $CH^i_\sim(X; \mathbb{Q})$ and in particular

$$p : CH^{i+m}_\sim(X; \mathbb{Q}) \to CH^{i+m}_\sim(X; \mathbb{Q})$$

Define $CH^i(M) := \mathrm{Im}(p) \subset CH^{i+m}_\sim(X; \mathbb{Q})$.

If $\sim$ is finer than or equal to homological equivalence, then p also operates on the the cohomology groups

$$p : H^{i+2m}(X_{\bar{k}}) \to H^{i+2m}(X_{\bar{k}})$$

Definition (Cohomology of motives)

For a motive $M = (X, p, m)$, the cohomology group of M is $H^i(M_{\bar{k}}) := \mathrm{Im}(p) \subset H^{i+2m}(X_{\bar{k}})$.

So we have a **realisation functor**:

$$R : \mathcal{M}_{\sim}(k) \to (\mathit{Vect}_F), M \mapsto H^*(M_{\bar{k}}).$$

At first glance, the realisation functor would dependent on Weil cohomology. Note that the category $\mathcal{M}_{num}(k)$ is defined intrinsically independent of any cohomology theory. Grothendieck emphasized it. But there is no realisation functor on $\mathcal{M}_{num}(k)$ unless the conjecture $D(X)$ is ture.

Grothendieck's Standard Conjecture

Now, let k be an algebraically closed field. We fix a Weil cohomology $H(X)$. Recall that we have the cycle class map

$$CH^i(X;\mathbb{Q}) \xrightarrow{\gamma_X} \mathrm{Im}(\gamma_X) = A^i(X) \subset H^{2i}(X)$$

Let $\Delta(X) \subset X \times_k X$ be the diagonal and consider its class

$$\gamma_{X\times_k X}(\Delta(X)) \subset H^{2d}(X \times_k X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X).$$

We write $\Delta_i^{topo} \in H^{2d-i}(X) \otimes H^i(X)$

Conjecture (The Künneth conjecture)

The Künneth components Δ_i^{topo} are algebraic, i.e. there is a cycle class $\Delta_i \in CH^d(X \times_k X;\mathbb{Q})$ such that $\gamma_{X\times_k X}(\Delta_i) = \Delta_i^{topo}$.

Let X_d be an irreducible smooth projective scheme, and let Y be a smooth hyperplane section. Let $\gamma_X(Y) \in A^1(X) \subset H^2(X)$ be its class and let

$$L : H^i(X) \to H^{i+2}(X), \alpha \cup \gamma_X(Y)$$

be the Lefschetz operator. We have the hard Lefschetz theorem

$$L^{d-i} : H^i(X) \xrightarrow{\sim} H^{2d-i}(X), 0 \leq i \leq d.$$

Consider the following maps

$$H^{d-j}(X) \underset{\sim}{\xrightarrow{L^j}} H^{d+j}(X) \xrightarrow{L} H^{d+j+2}(X) \underset{\sim}{\xleftarrow{L^{j+2}}} H^{d-j-2}(X)$$

for $i = d - j, 0 \leq j \leq d - 2$. We can define a unique map $\Lambda : H^i(X) \to H^{i-2}(X)$ making the resulting diagram commute. Similar for $i = d + 1, i = d + j, 2 \leq j \leq d$. The map Λ can be viewed as an element of $H^*(X)^\vee \otimes H^*(X) \simeq H^*(X) \otimes H^*(X) \subset H^*(X \times X)$.

Conjecture (Lefschetz type)

The map Λ is algebraic, i.e. $\Lambda = \gamma_{X \times_k X}(Z)$ for some $Z \in CH^{d-1}(X \times_k X; \mathbb{Q})$.

Define the i-th primitive cohomology

$$P^i(X) := ker(L^{d-i+1} : H^i(X) \to H^{2d-i+2}(X)).$$

We may speak of the primitive algebraic classes

$$A^i_{prim}(X) := A^i(X) \cap P^{2i}(X).$$

Cup product induces for $i \leq d/2$ a pairing

$$A^i_{prim}(X) \times A^i_{prim}(X) \to \mathbb{Q}, (x, y) \mapsto (-1)^i \mathrm{Tr} \circ (L^{d-2i}(x) \cup y).$$

Conjecture (Hodge type)

The pairing is positive definite.

One of the motivations of Grothendieck's standard conjecture is the Weil conjecture. Deligne proved the Weil conjecture without resorting the standard conjecture.
"If I had done it using motives, he would have been very interested, because it would have meant that the theory of motives had been developed. Since the proof used a trick, he did not care."-Pierre Deligne
The Hodge conjecture implies the standard conjecture over fields of characteristic zero, and whole standard conjecture together with the Tate conjecture.
On the contrary, the standard conjecture over $\mathbb{C}$ (or $\mathbb{F}_p$) implies the Hodge conjecture for abelian varieties (or the Tate conjecture for abelian varieties over $\mathbb{F}_p$).
Beilinson pointed out that existence of motivic t-structure on Voevodsky's motives implies the standard conjecture over fields of characteristic zero.

Remark

According to Illusie: Grothendieck asked John Coates to write down notes about his lectures on motives. Coates did it, but they were returned to him with many corrections. Coates was discouraged and quit. It was Kleiman who wrote down the notes.

Hodge theory: a brief recalling

The Hodge decomposition is concerning about compact Kähler manifolds. For our purpose, we restrict to smooth projective schemes over $\mathbb{C}$. Let X be a smooth projective scheme over $\mathbb{C}$. We have the following isomorphisms

$$H^i_{Zar}(X, \Omega^*_{X/\mathbb{C}}) \cong H^i_{Sing}(X(\mathbb{C}), \underline{\mathbb{C}}) \cong H^i_{dR}(X(\mathbb{C}), \Omega^*_{X(\mathbb{C})/\mathbb{C}})$$

We have the following spectral sequence

$$E_2^{p,q} = H^p_{Zar}(X, \Omega^q_X) \Rightarrow H^{p+q}_{Zar}(X, \Omega^*_{X/\mathbb{C}})$$

Theorem

The above spectral sequence is degenerate and we have the following decomposition

$$H^k(X(\mathbb{C}), \underline{\mathbb{C}}) \cong H^k(X, \Omega^*_X) \cong \bigoplus_{p+q=k} H^{p,q}(X) = \bigoplus_{p+q=k} H^q(X, \Omega^p_X).$$

A description

Denote by $A^k(X)$ the space of $\mathbb{C}$-valued differential k-forms on X^{an}. We have $A^k(X) = \bigoplus_{a+b=k} A^{a,b}(X)$. The term $A^{a,b}(X)$ locally looks like

$$dz_{i_1} \wedge \cdots \wedge dz_{i_a} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_b}$$

The complex

$$0 \to \Omega^p(X) \to A^{p,0}(X) \to A^{p,1}(X) \to \cdots \to A^{p,n}(X) \to 0$$

is exact.

Theorem

One gets an isomorphism

$$H^q(X, \Omega_X^p) \cong \frac{\ker(A^{p,q}(X) \to A^{p,q+1}(X))}{im(A^{p,q-1}(X) \to A^{p,q}(X))}$$

Let $Z \subset X$ be an integral closed subscheme of codimension p, and hence of dimension $n - p$. It determines a **cycle class**

$$[Z^{an}] \in H^{2p}(X^{an}, \mathbb{Z}(p))$$

as follows. Let $\tilde{Z}$ be a resolution of singularities of Z (it is possible by Hironaka) and let $\mu : \tilde{Z} \to X$ be the induced morphism. By Poincaré duality, the linear functional

$$H^{2n-2p}(X^{an}, \mathbb{Z}(n-p)) \to \mathbb{Z}, \alpha \mapsto \frac{1}{(2\pi i)^{n-p}} \int_{\tilde{Z}^{an}} \mu^*(\alpha)$$

is represented by a unique class $\xi \in H^{2p}(X^{an}, \mathbb{Z}(p))$ with property that

$$\frac{1}{(2\pi i)^{n-p}} \int_{\tilde{Z}^{an}} \mu^*(\alpha) = \frac{1}{(2\pi i)^{n}} \int_{X^{an}} \xi \cup \alpha.$$

The class $[Z^{an}]$ is of type (p, p). Indeed, if $\alpha \in H^{2n-2p}(X^{an}, \mathbb{Z}(n-p))$ is of type $(n-i, n-j)$ with $i \neq j$, then either i or j is strictly greater than p, and $\int_{\tilde{Z}^{an}} \mu^*(\alpha) = 0$.

Alternate construction of cycle map

Recall that $H^r_{Zar}(X, \mathcal{K}^M_{r,X}) \cong CH^r(X)$. The $d\log$ map

$$\mathcal{K}^M_{r,X} \to \Omega^r_X, \{f_1, \ldots, f_r\} \mapsto \wedge_j d\log f_j$$

induces a morphism of complexes in the Zariski topology $\{\mathcal{K}^M_{r,X} \to 0\} \to \Omega^{*\geq r}_X[r]$, and thus using GAGA

$$CH^r(X) \cong H^r_{Zar}(X, \mathcal{K}^M_{r,X}) = H^r(\{\mathcal{K}^M_{r,X} \to 0\}) \to H^r(\Omega^{*\geq r}_X[r]) = F^r H^{2r}_{dR}(X, \mathbb{C}).$$

This construction is purely algebraic.

Definition (Hodge classes)

Let $\varphi : H^{2p}(X(\mathbb{C}), \mathbb{Z}) \to H^{2p}(X(\mathbb{C}), \mathbb{C})$ be the canonical map and define the group of **integral Hodge classes** of type (p, p) as

$$H^{p,p}(X, \mathbb{Z}) := \{x \in H^{2p}(X, \mathbb{Z}) : \varphi(x) \in H^{p,p}(X, \mathbb{C})\}.$$

Similarly, the group of **rational Hodge classes** of type (p, p) is

$$H^{p,p}(X, \mathbb{Q}) := H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

Henceforth, every cycle class is a Hodge class.

Theorem (Lefschetz (1, 1)-theorem, 1924)

The map

$$cl : CH^1(X) \to H^{1,1}(X, \mathbb{Z})$$

is onto.

Proof.

The exponential exact sequence

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

gives a long exact sequence

$$\cdots \to H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1)) \xrightarrow{f} H^2(X, \mathcal{O}_X) \to \dots$$

The map f identifies with the composition

$$H^2(X, \mathbb{Z}(1)) \to H^2(X, \mathbb{C}) \to H^{0,2}(X) \cong H^2(X, \mathcal{O}_X)$$

So $\ker f$ is exactly the set of integral Hodge classes. A class $\alpha \in H^2(X, \mathbb{Z}(1))$ which maps to 0 in $H^2(X, \mathcal{O}_X)$ has $\alpha^{0,2} = 0$ in the Hodge decomposition. But the it also has $\alpha^{2,0} = \overline{\alpha^{0,2}} = 0$, and thus it is of type $(1, 1)$ hence a Hodge class. ■

Hodge's conjecture

Hodge conjectured that

$$cl : CH^p(X) \to H^{p,p}(X, \mathbb{Z})$$

is also surjective when $p \geq 2$. However, it is not true as pointed out by Atiyah and Hirzebruch.
The millennium problem is:

Conjecture

The rational Hodge classes are algebraic, i.e. the map

$$cl_{\mathbb{Q}} : \mathcal{Z}^p(X) \otimes \mathbb{Q} \to H^{p,p}(X, \mathbb{Q})$$

is surjective.

An equivalent form

The functor $X \mapsto X(\mathbb{C})$ is exact. Hence induces a natural transformation

$$\eta^0 : K_0 \to K^0_{top}$$

We have the Chern character isomorphisms:

$$K^0_{top}(X^{an}) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X, \mathbb{Q}), K_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_{p \geq 0} CH^p(X) \otimes \mathbb{Q}$$

Let X be a smooth projective scheme over $\mathbb{C}$. If E is a vector bundle over X. Then

$$c_p(E) \in F^p H^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{R}) \subset H^{2p}(X, \mathbb{C})$$

The Hodge conjecture is equivalent to the assertion that the image of η^0 is precisely equal to the subspace of all elements α such that $c_p(\alpha) \in F^p H^{2p}(X, \mathbb{C})$.

ℓ-adic cycle class map

Let $i : Y \to X$ be a closed embedding of equi-codimension r in X. The Gysin morphism induces a morphism

$$i_* : H^*(Y, \mathbb{Q}_\ell) \to H^{*+2r}(X, \mathbb{Q}_\ell(r))$$

The cycle class of Y is $i_*(1)$. Tate made the following analogue of the Hodge conjecture:

Conjecture

Tate, Beilinson Let k be a finitely generated field. Let X be a smooth, projective, and geometrically integral scheme over K. The cycle class homomorphism

$$cl : \mathcal{Z}^r(X) \otimes \mathbb{Q}_\ell \to H^{2r}(X_{k_s}, \mathbb{Q}_\ell(r))^{Gal(k_s/k)}$$

is surjective.
If k is a finite field, Beilinson conjectured cl is also injective.

The Hodge and the Tate conjectures are not unrelated.

Theorem (Paranjape-Sapiro)

For an abelian variety X defined over a finitely generated subfield $L \subset \mathbb{C}$, the Tate conjecture implies the Hodge conjecture.

Theorem (Milne)

If the Hodge conjecture holds for all abelian varieties of CM type over $\mathbb{C}$, then the Tate conjecture holds for all abelian varieties defined over the algebraic closure of a finite field.

Voevodsky's derived category of mixed motives

Definition

The category of finite correspondence Cor_k is an additive category with objects smooth k-schemes. Morphism X to Y are finite linear combinations of correspondences $Z = \sum n_i Z_i$ where $Z_i \in X \times_k Y$ is closed and the projection $\pi_i : Z_i \to X$ is finite and surjective.

There is a functor $\Gamma : Sm/k \to Cor_k$ which is identity on objects, maps a morphism to its graph.
For $X \in Sm/k$, the representable sheaf $\mathbb{Z}_{tr}(X)$ defined by $\mathbb{Z}_{tr}(X)(U) := \mathrm{Hom}_{Cor_k}(U, X)$.

Definition

Voevodsky's triangulated category of mixed motives is defined as $DM^{eff}_{Nis}(k) := D^{-}Sh_{Nis}(Cor_k)[W^{-1}]$ where W is the smallest thick subcategory containing all cones of $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$.

The motive associated to a smooth k-scheme X is defined by $M(X) := \mathbb{Z}_{tr}(X) \in DM^{eff}_{Nis}(X)$. We write $\mathbb{Z}(1) = M(\mathbb{G}_m)[-1]$

Definition

Voevodsky's motivic cohomology is defined by

$$H^p_M(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM^{eff}_{Nis}(k)}(M(X), \mathbb{Z}(q)[p]).$$

We can also change the Nisnevich topology to étale topology and coefficients ring $\mathbb{Z}$ to $\mathbb{Z}[1/\mathrm{char}(k)]$ get the étale motives and the corresponding motivic cohomology is called étale motivic cohomology, denoted by $H^{p,q}_L$.

Theorem

Étale motivic cohomology with finite coefficients is just étale cohomology with finite coefficients.

Theorem

Rational motivic cohomology equal with rational étale motivic cohomology.

The following isomorphism is highly non-trivial:

Theorem (Voevodsky-Rost)

Let K be a field and $\ell \in k^\times$. The norm residue homomorphism

$$K_n^M(k)/\ell \to H_{ét}^n(k; \mu_\ell^{\times n})$$

is an isomorphism.

It is equivalent to for every smooth X over K and all $p \leq n$, the motivic to étale map

$$H_M^p(X, \mathbb{Z}/\ell(n)) \to H_{\text{ét}}^p(X, \mu_\ell^{\otimes n})$$

is an isomorphism.

Here are some applications of the Voevodsky-Rost theorem.

Theorem

For any integer $k > 0$,

$$K_{4k+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2^{(k+1) \bmod 2}\mathbb{Z}.$$

[Lichtenbaum conjecture] Let F *be a totally real field with* r_1 *real embeddings. Then*

$$\zeta_F(1-2k) = (-1)^{kr_1} \frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|}$$

up to power of 2.

$$\frac{(-1)^k}{2}\zeta(1-2k) = \frac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|}$$

Conjecture (Consequence of the Tate conjecture)

If X is smooth and projective over a finite field, we have $H^i_M(X, \mathbb{Q}(n)) = 0$ for any $i \neq 2n$.

Conjecture (Soulé)

For X regular and proper dimension d over $\mathbb{Z}$. We have

$$\sum (-1)^i \dim_{\mathbb{Q}} H^i_M(X, \mathbb{Q}(p)) = -ord_{s=d-p}\zeta_X(s)$$

the right hand side is the negative of the order of zero or poles at $s - d - p$ of the zeta function of X.

It implies part of the BSD conjecture!

Higher Chow group

Define the simplex Δ^n by

$$\Delta^n := \mathrm{Spec}(k[t_0, \dots, t_n]/(\sum t_i - 1)).$$

Notice that Δ^n is isomorphic to $\mathbb{A}^n_k$. By setting the coordinates $t_i = 0$, one obtains $(n+1)$ linear hypersurfaces in Δ^n called the **codimension one faces**. As usual, the points $p_i = (0, \dots, 1, \dots, 0)$ are called the vertives of the simples Δ^n.
By iterating this, one gets docimension $(n-m)$-faces isomorphic to Δ^m for every $m < n$.
Let $Z^p(X, n) \subset Z^p(X \times \Delta^n)$ be the subset of cycles Z of $X \times \Delta^n$ of codimension p such that every irreducible component of Z meets all faces $X \times \Delta^m$ properly.
Let $\partial_i : Z^p(X, n) \to Z^p(X, n-1)$ be the restriction map to i-th codimension one face for $i = 0, \dots, n$ and let $\partial = \sum (-1)^i \partial_i$. We have the map ∂ is a differential of degree -1 and endows the family $Z^p(X, *)$ with a structure of homological complex of abelian groups.

Definition

The n-th homology group of the complex

$$\cdots \to Z^p(X, n+1) \xrightarrow{\partial} Z^p(X, n) \xrightarrow{\partial} Z^p(X, n-1) \to \dots$$

is called the n-th higher Chow group, denoted by $CH^p(X, n)$.

Theorem (Voevodsky)

$$H^n_M(X, \mathbb{Z}(i)) \cong CH^i(X, 2i - n).$$