A brief introduction to Hitchin's equations

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Classical Hodge Theory

Classical Hodge Theory

Support X is a compact Kähler manifold, then we have

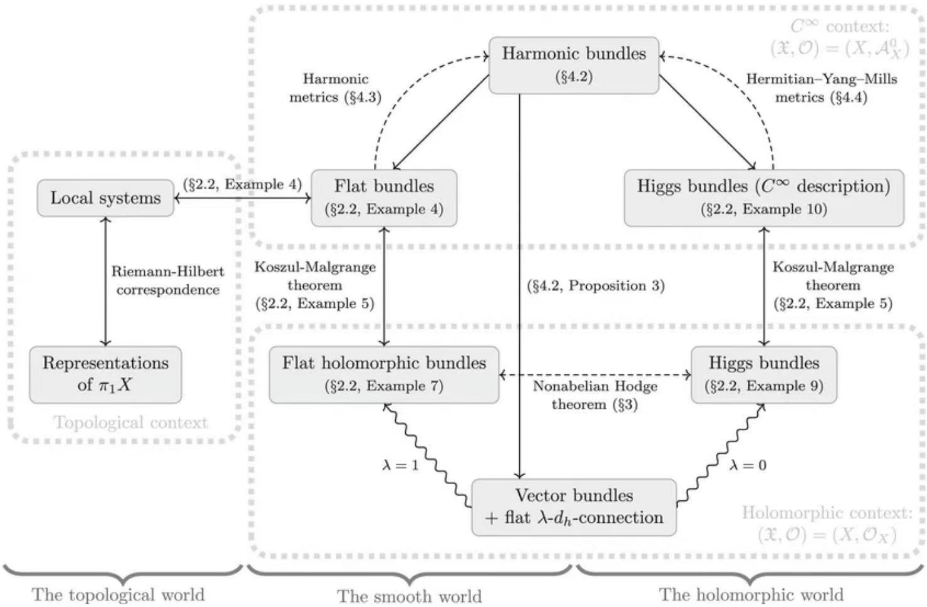
$$Hom(\pi_1 X, \mathbb{C}) \simeq H^1_{Sing}(X, \mathbb{C}) \simeq H^1_{dR}(X, \mathbb{C}) \simeq \oplus_{p+q=1} H^{p,q}(X).$$

- Classical Hodge theory actually builds a bridge connecting topology, differentiable and holomorphic world.

Note $\mathbb{C}$ is an abelian group, if we replace $\mathbb{C}$ with a nonabelian group, will we have a similar result?

Non-abelian Hodge Theory

From: Introduction to Nonabelian Hodge Theory



The objects in nonabelian Hodge theory and their relationships

Figure 1: From R. Laza, Matthias Schuett, and Noriko Yui. Calabi-Yau Varieties: Arithmetic, Geometry and Physics, Lecture Notes on Concentrated Graduate Courses. 01 2015.

Motivitions

Hitchin's equations appear as a dimensional reduction of the self-dual Yang–Mills equations from four dimension to two dimension that have the form:

$$\begin{cases} F_{D_{E,h}} + [\theta, \theta^\dagger] = 0 \\ \overline{\partial}_E \theta = 0 \end{cases}$$

- The irreducible solutions to Hitchin's equations modulo gauge transformations are actually the same as stable Higgs pairs. This relates something purely algebro-geometric and something analytic.
- The moduli space of the irreducible solutions has rich geometric structures. As for its topology property, it is non-compact,simply connected and has dimension $12(g-1)$. From the analytical aspect, it has a natural metric which is hyperKähler. Moreover, some explicit solutions can be used to construct hyperbolic metric and give examples to Higgs bundles.

Contents

Basic notions

Through this talk, we use the following notations.

- Σ: a Riemann surface with genus g greater than 1;
- E : complex vector bundle of rank r;
- $\Omega^p(\Sigma, E)$: $\mathbb{C}^\infty$ complex p-forms valued in E;
- $\Gamma(\Sigma, E)$: the space of smooth sections of E;
- K: the canonical line bundle $(T^{1,0})^*\Sigma$.

Hermitian vector bundle

Holomorphic structure

A holomorphic structure on E over Σ is a differential operator $\overline{\partial_E} : \Omega^{p,q}(\Sigma, E) \to \Omega^{p,q+1}(\Sigma, E)$ satisfying the Leibniz rule: for any $\alpha \in \Omega^{p,q}(\Sigma, \mathbb{C}), \beta \in \Omega^{k,l}(\Sigma, E)$,

$$\overline{\partial_E}(\alpha \wedge \beta) = (\overline{\partial}\alpha) \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial_E}\beta.$$

Hermitian metric

A Hermitian metric H on a complex vector bundle over Σ is a $\mathbb{C}$ family of Hermitian inner products on E satisfying the following conditions:

- $H(x, y)$ is linear in x, where x, y is in the fiber E_p for some p;
- $H(x, \overline{y}) = \overline{H(x, \overline{y})}$;
- $H(x, x) > 0$ for any $x \neq 0$.

Connection in vector bundles

Hermitian vector bundle

If there exists a Hermitian metric on a vector bundle E, then (E, H) is called a Hermitian vector bundle.

Connection

A connection D on a complex vector bundle E is a differential operator from $\Omega^p(\Sigma, E)$ to $\Omega^{p+1}(\Sigma, E)$ such that for any $f \in \Omega^p(\Sigma, \mathbb{C}), g \in \Omega^p(\Sigma, E)$, $D(f \wedge g) = Df \wedge g + (-1)^p f \wedge dg$.

Connection in vector bundles

Curvature

The curvature F_D of connection D is defined to be the operator

$$F_D = D \circ D : \Omega^p(\Sigma, E) \to \Omega^{p+2}(\Sigma, E).$$

Flat connection

A connection is said to be flat if $F_D = 0$.

Unitary connection

A connection D on E is unitary if for any two sections $s, t \in \Gamma(\Sigma, E)$,

$$d(H(s,t)) = H(Ds, t) + H(s, Dt),$$

where H is a Hermitian metric on E.

Chern connection

Theorem

For a holomorphic vector bundle E with a Hermitian metric H, there exists a unique connection $\nabla_{\overline{\partial_E},H}$, called Chern connection, such that $\nabla^{0,1}_{\overline{\partial_E},H} = \overline{\partial_E}$ and $\nabla_{\overline{\partial_E},H}$ is a unitary connection.

Higgs bundles

A rank r Higgs bundle over a Riemann surface Σ is a pair (E, ϕ), where E is holomorphic vector bundle of rank r, and $\phi \in \Omega^{1,0}(\Sigma, EndE \otimes K)$ which is called the Higgs field.

Stable Higgs bundles

First Chern class

The first Chern class of E is $c_1(E) = [\frac{i}{2\pi} tr(F_D)] \in H^2_{dR}(\Sigma, \mathbb{C})$.

Degree

The degree of the vector bundle E is $degE = \int_\Sigma c_1(E)$.

Slope

The slope of the vector bundle E is $\mu(E) = \frac{degE}{rank(E)}$.

Stable Higgs bundles

Stable Higgs bundle

A Higgs bundle (E, ϕ) is stable if for every proper, non-trivial, ϕ-invariant subbundles $F \subset E$, $\mu(F) < \mu(E)$.
We call (E, ϕ) semistable if $\mu(F) \leqslant \mu(E)$.

Example

Let $E = K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}$,where $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} = K$, then $\mu(E) = 0$.
Take $\phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, where 1 denotes the section of $Hom(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \otimes K$.
The only ϕ-invariant subbundle is $K^{-\frac{1}{2}}$, which has degree $1 - g < 0$ for $g > 1$, so (E, ϕ) is a stable Higgs bundle.

Hermitian-Yang-Mills equation

Definition

Let (E, h) be a trivial Hermitian vector bundle over $\mathbb{R}^4$, and ∇ a unitary connection on (E, h). Then the curvature F_∇ of ∇ is said to satisfy the self-dual Yang-Mills equations if F_∇ is invariant under the Hodge star operator $*$, that is,

$$*F_\nabla = F_\nabla.$$

In terms of a trivialization of (E, h) over $\mathbb{R}^4$ and the basic coordinates (x_1, x_2, x_3, x_4). We may write

$$F_{\nabla} = \frac{1}{2} \sum F_{ij} dx_i \wedge dx_j,$$

then the equation is reduced to the followings:

$$F_{12} = F_{34},\ F_{13} = F_{42},\ F_{14} = F_{23}.$$

We can write $\nabla = d + \sum_i A_i dx_i$, where $A_i \in U(n)$.
Now we assume ∇ is independent of x_3, x_4, and set $\phi_1 = A_3,\ \phi_2 = A_4$.
Define $\nabla_i = \frac{\partial}{\partial x_i} + A_i$ for $i \in \{1, 2\}$, then the equation can written as

$$[\nabla_1, \nabla_2] = [\phi_1, \phi_2], [\nabla_1, \phi_1] = [\phi_2, \nabla_2], [\nabla_1, \phi_2] = [\nabla_2, \phi_1].$$

From Hermitian-Yang-Mills equation to Hitchin's equation

Define a connection $D = d + A$ on E by $A = A_1 dx_1 + A_2 dx_2$ and let $\phi = \phi_1 - i\phi_2, D_j = \frac{\partial}{\partial x_j} + A_j$ for $j = 1, 2$.
Then we obtain that

$$F_D = \frac{i}{2}[\phi, \phi^\dagger], [D_1 + iD_2, \phi] = 0.$$

Here the dagger $\dagger$ stands for the transpose conjugate.
In order to identify $\mathbb{R}^2$ with $\mathbb{C}$, set

$$z = x_1 + ix_2, \partial_{\overline{z}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}, A_{\overline{z}} = A_1 + iA_2,$$

then write

$$D_1 + iD_2 = D_{\overline{z}} = \partial_{\overline{z}} + A_{\overline{z}}$$

Let $\theta = \frac{1}{2}\phi dz, \theta^\dagger = \frac{1}{2}\phi^\dagger d\overline{z}$, the equation can be written as follows:

$$\begin{cases} F_D + [\theta, \theta^\dagger] = 0 \\ D_{\overline{z}}\theta = 0 \end{cases}$$

Hitchin's equations

Hitchin's equations

Given a Hermitian holomorphic vector bundle $(E, \overline{\partial}_E, h)$ on a Riemann surface Σ, $\phi \in \Omega^{1,0}(\Sigma; EndE)$. Let $D_{E,h}$ be the Chern connection of $\overline{\partial}_E$ with respect to h, $(D_{E,h}, \phi)$ is said to satisfy the Hitchin's equations if

$$\begin{cases} F_{D_{E,h}} + [\phi, \phi^{\dagger}] = 0 \\ \overline{\partial}_E \phi = 0 \end{cases}$$

Non-abelian Hodge correspondence

Theorem(Hitchin and Simpson)

There is a correspondence between stable Higgs bundles and irreducible solutions of Hitchin's equations.

$$\{stable\ Higgs\ bundles\ \}/\sim\ \leftrightarrow$$

$$\{irreducible\ solutions\ to\ Hitchin's\ equations\ \}/\sim$$

- Remark: Furthermore, if we consider harmonic representation we can obtain another one-to-one correspondence making the Theorem into the so-called non-abelian Hodge correspondence.

$$\{stable\ Higgs\ bundles\ (E,\phi)\ of\ rank\ n\}/\sim\ \leftrightarrow$$

$$\{irreducible\ projectively\ flat\ connections\}/\sim\ \leftrightarrow$$

$$\{Representations\ of\ \pi_1\Sigma\ into\ SL(n,\mathbb{R})\}/\sim$$

Example

Example

Let $\phi = 0$, then the equations are simply $F_D = 0$, so it is reduced to the flat unitary connections on Σ, which are equivalent to stable holomorphic bundles of Σ by Narasimhan-Seshadri Theorem.

Narasimhan-Seshadri Theorem

$$\{flat\ unitary\ connections\}/\!\sim\ \leftrightarrow$$

$$\{stable\ holomorphic\ vector\ bundle\}/\!\sim$$

Connection to hyperbolic geometry

Theorem(Hitchin)

Let $a \in H^0(\Sigma; K^2)$ be a quadratic differential, (D, ϕ) be a solution to the Hitchin's equations for which $E = K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}$ and $\phi = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$, then

- If h is the Hermitian metric on K^{-1}, then $\hat{h} = a + (h + \frac{a\overline{a}}{h}) + \overline{a}$ is a Riemannian metric on Σ of constant curvature -4.
- Any metric of constant curvature -4 on Σ is isometric to a metric of this form for some $a \in H^0(\Sigma; K^2)$.

Metric on moduli space of the solutions

Theorem(Hitchin)

Let $\mathcal{M}$ be the moduli space of irresucible solutions to the Hitchin's equations on a rank 2 vector bundle V of odd degree. Then the natural metric is complete and hyperKähler.

HyperKähler metric

A hyperKähler metric on a $4n$-dimensional manifolds is a Riemannian metric which is Kählerian with respect to three almost complex structures I, J, K which satisfy the following equations:

$$\begin{cases} I^2 = J^2 = K^2 = -1, \\ IJ = -JI = K, \\ JK = -KJ = I, \\ KI = -IK = J. \end{cases}$$

Topology information of the moduli space

Dimension

Let $\mathcal{M}$ be the moduli space of irreducible solutions to Hitchin's equations on a rank 2 vector bundle of odd degree V, then $\mathcal{M}$ is a smooth manifold of dimension $12(g-1)$.

From Higgs bundle aspect

The moduli space of all stable pairs (V, ϕ), where V is a rank 2 holomorphic vector bundle of odd degree and fixed determinant, ϕ is a trace-free holomorphic section of $EndV \otimes K$, is a smooth manifold of real dimension $12(g-1)$.

References

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