The Space of Persistence Diagrams on n Points Coarsely Embeds into Hilbert Space

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Preliminary

Definition

(1) metric d_∞ on $\mathbb{R}^2$ by

$d_\infty\left(\left(x_1, x_2\right), \left(y_1, y_2\right)\right) = \max\left\{\left|x_1 - y_1\right|, \left|x_2 - y_2\right|\right\}$;

(2) $\mathcal{D}^1 = T \cup \{\Delta\}$ where $\Delta \notin T = \left\{\left(x_1, x_2\right) \in \mathbb{R}^2 \mid x_2 > x_1 \geq 0\right\}$;

(3) semi-metric δ on $\mathcal{D}^1$ as an extension of $d_\infty|_T$ on T by defining

$\delta\left(\left(x_1, x_2\right), \Delta\right) = \left(x_2 - x_1\right)/2$.

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Spaces of persistence diagrams

Choose $n \in \mathbb{N}$. Define spaces of persistence diagrams:

Definition

(1) **the space of persistence diagrams** on at most n points as $\mathcal{D}^n = \left(\mathcal{D}^1\right)^n / \mathcal{S}_n$, where the group of symmetries $\mathcal{S}_n$ acts on the coordinates by permutation, i.e., we identify diagrams $z = (z_1, z_2, \ldots, z_n), z' = (z'_1, z'_2, \ldots, z'_n) \in \left(\mathcal{D}^1\right)^n$ iff there exists a matching φ on $\{1, 2, \ldots, n\}$ so that $z_i = z'_{\varphi(i)}$

(2) a natural inclusion $\mathcal{D}^n \subset \mathcal{D}^{n+1}$ by appending point Δ. We will frequently use this inclusion implicitly, for example by identifying diagrams (a) and (a, Δ). Consequently we can define $\mathcal{D}^{<\infty} = \bigcup_{n \in \mathbb{N}} \mathcal{D}^n$.

Bottleneck distance

Definition

- For points $z = (z_1, z_2, \ldots, z_n), z' = (z'_1, z'_2, \ldots, z'_n)$ in $\mathcal{D}^n$ define
  - $A(z) = (z_1, z_2, \ldots, z_n, \Delta, \ldots, \Delta) \in \left(\mathcal{D}^1\right)^{2n}$
  - $A(z') = (z'_1, z'_2, \ldots, z'_n, \Delta, \ldots, \Delta) \in \left(\mathcal{D}^1\right)^{2n}$

  The bottleneck distance is defined as

$$d_{\mathcal{B}}(z, z') = \min_{\varphi \in \mathcal{S}_{2n}} \max_i \delta\left(z_i, z'_{\varphi(i)}\right)$$

- $\mathcal{D}^n_{\mathcal{B}} = (\mathcal{D}^n, d_{\mathcal{B}})$ and $\mathcal{D}^{<\infty}_{\mathcal{B}} = (\mathcal{D}^{<\infty}, d_{\mathcal{B}})$. Note that $\mathcal{D}^1_{\mathcal{B}}$ is not isometric to $\left(\mathcal{D}^1, \delta\right)$.

p-Wasserstein distance

Definition

- For points $z = (z_1, z_2, \dots, z_n), z' = (z'_1, z'_2, \dots, z'_n)$ in $\mathcal{D}^n$

  The p-Wasserstein distance is defined as

$$d_{\mathcal{W},p}(z, z') = \min_{\varphi \in \mathcal{S}_{2n}} \left(\sum_i \delta\left(z_i, z'_{\varphi(i)}\right)^p \right)^{1/p}$$

- $\mathcal{D}^n_{\mathcal{W},p} = (\mathcal{D}^n, d_p)$ and $\mathcal{D}^{<\infty}_{\mathcal{W},p} = (\mathcal{D}^{<\infty}, d_p)$

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Background

Coarse embeddings were once in the spotlight due to a remarkable theorem by G. Yu [8], who showed that every discrete metric space Γ which embeds coarsely into a Hilbert space satisfies the Coarse Baum-Connes Conjecture. In particular, if Γ is a finitely generated group with word length metric coarsely embeddable into a Hilbert space and the classifying space $B\Gamma$ has a homotopy type of a finite CW-complex, then the Novikov Conjecture holds for Γ.

Coarse Structure

Let X be a set. The **product** of two sets $A, D \subset X^2$, denoted $A \circ D$ is given by

$$A \circ D = \left\{(x,y) \in X^2 \mid \exists z \in X \ni (x,z) \in A, (z,y) \in D\right\}$$

Definition

Let X be a set. A **coarse structure** on X is a collection of subsets of $\mathcal{C} \subseteq \mathcal{P}(X^2)$ satisfying $\Delta \in \mathcal{C}$ in addition to the following four closure properties:

1. $A \in \mathcal{C} \Rightarrow D \in \mathcal{C}$ for any $D \subseteq A$ (closed under subsets)
2. $A \in \mathcal{C} \Rightarrow A^t \in \mathcal{C}$ (closed under transpositions)
3. $A, D \in \mathcal{C} \Rightarrow A \cup D \in \mathcal{C}$ (closed under finite unions)
4. $A, D \in \mathcal{C} \Rightarrow A \circ D \in \mathcal{C}$ (closed under finite products)

A **coarse space** is a set X endowed with a coarse structure $\mathcal{C}$. The sets in $\mathcal{C}$ are called **controlled sets**. Any subset B of X for which B^2 is controlled is called bounded. A coarse space is **connected** if every point $(x, y) \in X^2$ lies in some controlled set.

Example

Let X be a set. The first (trivial) example of a coarse structure on X is the power set of X^2, called the **maximal coarse structure**. Another is the collection $\mathcal{C}_{\text{dis}}$ consisting of all sets containing only finite many points off the diagonal Δ; this is called the **discrete coarse structure** on X. The discrete coarse structure is the smallest, connected coarse structure on X. Perhaps the most fundamental nontrivial example of a coarse space is a metric space (X, d) endowed with the **bounded coarse structure**. This is the structure consisting of all sets $\mathcal{C}$ such that

$$\sup\{d(x, y) \mid (x, y) \in \mathcal{C}\} < \infty$$

Uniformly bounded

Let X be a coarse space. We say a collection of bounded sets $\{B_\alpha\}$ is **uniformly bounded** if $\bigcup B_\alpha^2$ is controlled. The definition comes from [7]

Asymptotic dimension

Definition

Let n be a non-negative integer. We say that the asymptotic dimension of a metric space X is less than or equal to n ($\mathrm{asdim} X \leq n$) iff for every $R > 0$ the space X can be expressed as the union of $n + 1$ subsets X_i, with each X_i being an union of uniformly bounded R-disjoint sets.

The definition comes from [2]

$\text{asdim}(\mathbb{R}) \leq 1$

Example

To see that $\text{asdim}(\mathbb{R}) \leq 1$ we need to express $\mathbb{R}$ as the union of two families of uniformly bounded sets. See Figure 1 for a decomposition of $\mathbb{R}$ into two such families. Here $\mathbb{R}$ is endowed with the bounded coarse structure, the metric being the Euclidean metric.

Figure: $\text{asdim}\mathbb{R} \leq 1$

Coarse embedding and Coarse equivalence

Definition

Let $f : X \to Y$ be a function between metric spaces.

1. f is is said to be a coarse embedding if for $i = 1, 2$ there are non-decreasing functions $\rho_i : [0, \infty) \to [0, \infty)$ with $\rho_1(d(x_1, x_2)) \leq d(f(x_1), f(x_2)) \leq \rho_2(d(x_1, x_2))$ and with $\lim_{t\to\infty} \rho_1(t) = \infty$.
2. If, in addition, f is coarsely onto then f is said to be a coarse equivalence. A function $f : X \to Y$ is said to be coarsely onto if there is $D > 0$ such that the D-neighborhood of $f(X)$ is all of Y (for every $y \in Y$ there is $x \in X$ such that $d(f(x), y) \leq D$).

Asymptotic dimension is a coarse invariant

Let $f: X \to Y$ be a coarse equivalence. Then $\operatorname{asdim} X = \operatorname{asdim} Y$.

Proof.

If $\mathcal{U}^0, \ldots, \mathcal{U}^n$ are r-disjoint, D-bounded families covering X then the families $f(\mathcal{U}^i)$ are $\rho_1(r)$-disjoint and $\rho_2(D)$-bounded. Since $N_R(f(X))$ contains Y we see that taking families $N_R(f(\mathcal{U}^i))$ will cover Y and be $(2R + \rho_2(D))$-bounded and $(\rho_1(r) - 2R)$-disjoint. Since $\rho_i \to \infty$, r can be chosen large enough for $\rho_1(r) - 2R$ to be as large as one likes. Therefore, $\operatorname{asdim} Y \leq \operatorname{asdim} X$. The same proof applied to a coarse inverse for f proves that $\operatorname{asdim} X \leq \operatorname{asdim} Y$. □

Union Theorem and Product Theorem of asymptotic dimension

Theorem

Suppose X and Y are subspaces of a metric space Z. Then the following hold:

Union Theorem $\mathrm{asdim}(X \cup Y) = \max\{\mathrm{asdim}X, \mathrm{asdim}Y\}$ *[2, Corollary 26].*

Product Theorem $\mathrm{asdim}(X \times Y) \leq \mathrm{asdim}X + \mathrm{asdim}Y$ *[2, Theorem 32].*

Topological dimension

Recall that the multiplicity of a cover of a metric space is the maximum number of elements of the cover that can intersect. We will use this to define tpological dimension for lower bounds of asymptotic dimension. One can refer to [4] for details.

Topological dimension

Definition

1. Let n be a non-negative integer. We say that the topological dimension of a topological space X is less than or equal to n $(\dim X \leq n)$ iff for every open cover $\mathcal{U}$ of the space X there is an open cover $\mathcal{V}$ of X of multiplicity less than or equal to $n+1$.
2. If X is a compact metric space, the above definition is equivalent to the following: $\dim X \leq n$ iff for every $\varepsilon > 0$, X has an ε-small open cover of multiplicity $n+1$.

An important Lemma

Lemma

Let $p > 1$. *If for every* $R > 0$ *there is an isometric embedding of* $([0,R]^m, d_\infty)$ *or* $([0,R]^m, d_p)$ *in* X, *then* $\mathrm{asdim} X \geq m$.

One can refer to [Atish et al. [1] 2021 Lemma 2.12] for proof of this lemma.

An interesting result

For each $p > 1$ spaces $\mathcal{D}_{\mathcal{B}}^{<\infty}$ and $\mathcal{D}_{\mathcal{W},p}^{<\infty}$ are not of finite asymptotic dimension.

Proof.

For each $R > 0$ and $n \in \mathbb{N}$ we can isometrically embed $([0,R]^n, d_\infty)$ or $([0,R]^n, d_p)$ into $\mathcal{D}_{<\infty}^{n}$ or $\mathcal{D}_{p}^{<\infty}$ respectively by mapping $(x_1, x_2, \ldots, x_n) \mapsto (2R, 4R + x_1, 4R, 6R + x_2, \ldots, 2nR, 2nR + 2R + x_n)$. The conclusion follows by Lemma above. □

A classical result of coarse embedding

Finiteness of asymptotic dimension is closely related to embeddability questions, as the following well known result shows.

Theorem

[Roe, [7] Example 11.5] A metric space of finite asymptotic dimension coarsely embeds in Hilbert space.

Asymptotic dimension is an invariant of finite group action

Theorem

[Kasprowski, [5] Theorem 1.1] Let X be a proper metric space and F be a finite group acting on X by isometries. Then X/F has the same asymptotic dimension as that of X.

Coarse disjoint union

Given a sequence of bounded metric spaces (X_n, d_n) we can define a metric d on their disjoint union $\bigsqcup_n X_n$ such that d restricted to X_n is d_n, and for $i \neq j$ and $x_i \in X_i$, $x_j \in X_j$, $d(x_i, x_j) > \max\{\text{diam}(X_i), \text{diam}(X_j)\}$. Any two such metrics on $\bigsqcup_n X_n$ are coarsely equivalent and the resulting space is called **coarse disjoint union**.

In Theorem below we will consider $\mathbb{Z}_k = \{[0], \cdots, [k-1]\}$, the set of integers modulo $k \in \mathbb{N}$, as a metric space. The metric is defined as $d([i],[j]) = \min\{|i' - j'| : [i' - j'] = [i - j]\}$. This is the usual word metric on the finitely generated group $\mathbb{Z}_k$.

The application of asymptotic dimension in finitely generated group

Theorem

[Dranishnikov et al, [3] Proposition 6.3]

Consider $(\mathbb{Z}_n)^m$ as a metric space, where the integers mod n has the word metric and the m-fold product has the max metric d_∞. Let S be the disjoint union of $(\mathbb{Z}_n)^m$ (for all $m, n \geq 1$). We define a metric d on S whose restriction to each $(\mathbb{Z}_n)^m$ coincides with its existing metric, and such that $d(x, y) > m + n + m' + n'$ for $x \in (\mathbb{Z}_n)^m$ and $y \in (\mathbb{Z}_{n'})^{m'}$. Then S does not coarsely embed in a Hilbert space.

Finite determination

Often, an efficient way to decide coarse embeddability (and non-embeddability) of metric spaces is the following result, which says that this question is "finitely determined".

Theorem

[Nowak, [6] Theorem 3.4]

A metric space (X,d) *admits a coarse embedding in a Hilbert space if and only if for* $i = 1,2$ *there are non-decreasing functions* $\rho_i : [0,\infty) \to [0,\infty)$ *with* $\lim_{t\to\infty} \rho_1(t) = \infty$*, such that for every finite subset* $A \subset X$ *there exists a map* $f_A : A \to \ell_2$ *satisfying* $\rho_1(d(x_1,x_2)) \leq \|f_A(x_1) - f_A(x_2)\|_2 \leq \rho_2(d(x_1,x_2))$ *for all* $x_1, x_2 \in X$.

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Connection between the bottleneck distance ($\mathcal{D}_{\mathcal{B}}^n$) and the p-Wasserstein distances ($\mathcal{D}_{\mathcal{W},p}^n$)

For each $n \in \mathbb{N}$ and $p \geq 1$, $\mathcal{D}_{\mathcal{B}}^n$ and $\mathcal{D}_{\mathcal{W},p}^n$ are coarsely equivalent.

Proof.

This can be checked by direct comparison of the definitions of these metrics. □

Compute Asymptotic Dimension of Spaces of Persistence Diagrams with at most n points

The main result of this section is the following. Here we will provide a detailed proof for the case when n=1. The more general case can be completed using induction. The proof in the original text is rather technical, and one may refer to (Atish et al. [1] Theorem 3.2).

Theorem

For $n \in \mathbb{N}$*,* $\mathrm{asdim}\mathcal{D}^n_{\mathcal{W},p} = \mathrm{asdim}\mathcal{D}^n_{\mathcal{B}} = 2n$*.*

The easiest case

The following lemma deals with the case $n = 1$.

Lemma

$\mathrm{asdim}\mathcal{D}_{\mathcal{B}}^{1} = 2$

One may refer to [1].

Upper bounds

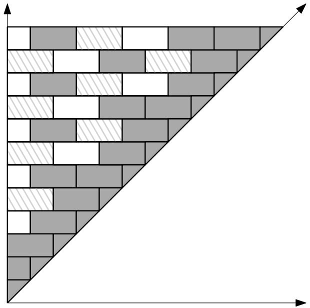

Figure: $\text{asdim}\mathcal{D}^1_{\mathcal{B}} \leq 2$

Lower bounds

We now turn attention to inequality $\mathrm{asdim}\mathcal{D}^1_{\mathcal{B}} \geq 2$. Given $R > 0$ the subset $\widetilde{B} = [0, R] \times [2R, 3R]$ in $\mathcal{D}^1_\infty$ is isometric to $([0, R]^2, d_\infty)$. To verify this note that $\widetilde{B}$ is of diameter R and at distance $2R$ from the diagonal, hence no optimal matching used when computing the induced distance on $\widetilde{B}$ pairs any point of $\widetilde{B}$ to the diagonal.

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Lemma

For each $h \in \mathbb{N}$ every finite metric space (X, d) embeds isometrically into $\mathcal{D}_{\mathcal{B}}^{|X|}$ above the horizontal line at height h.

Proof.

Let $X = \{x_0, x_1, \ldots, x_n\}$ and $R = \operatorname{diam}(X)$. For each k define

$$f(x_k) = \Big\{ \big(3Ri, 3Ri + 3R + d(x_k, x_i) + h\big) \mid i = 1, 2, \ldots, n \Big\}.$$

Note that $f \colon X \to f(X) \subset \mathcal{D}_{\mathcal{B}}^{|X|}$ is an isometry. For precise proof, one can refer to [Atish et al. [1] Lemma 4.1] □

Corollary

A coarse disjoint union of any collection of finite metric spaces $\{A_i\}_{i\in\{1,2,\ldots\}}$ *embeds isometrically into* $\mathcal{D}_{\mathcal{B}}^{<\infty}$*. If for some* $M \in \mathbb{N}$ *we have* $|A_i| \leq M, \forall i$*, then the embedded space lies within* $\mathcal{D}_{\mathcal{B}}^{M}$*.*

Proof.

Using Lemma above we can isometrically embed each A_i into $\mathcal{D}_{\mathcal{B}}^{|A_i|}$ above any height of our choosing. Starting with A_1 we inductively embed A_i into $\mathcal{D}_{\mathcal{B}}^{<\infty}$ using Lemma above so that the y-coordinates of the embedded A_i are at least $\max_{j\leq i} \operatorname{diam}(A_j)$ above the maximal y-coordinates of the embedded A_{i-1}. Let $\widetilde{A}$ denote the embedded union of $\{A_i\}_{i\in\{1,2,\ldots\}}$. The subspace metric on $\widetilde{A}$ turns $\widetilde{A}$ into a coarse disjoint union of $\{A_i\}_{i\in\{1,2,\ldots\}}$. If for some $M \in \mathbb{N}$ we have $|A_i| \leq M, \forall i$, then the embedding above maps each A_i into $\mathcal{D}_{\mathcal{B}}^{M}$ by Lemma above, and hence $\widetilde{A} \subset \mathcal{D}_{\mathcal{B}}^{M}$. □

Theorem

$\mathcal{D}_{\mathcal{B}}^{<\infty}$ does not coarsely embed into Hilbert space.

Proof.

Follows from Theorem of Dranishnikov above and last Corollary for the coarse disjoint union of $((\mathbb{Z}/m)^n, d_\infty)$. □

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