Toric Variety

Chunshuang Yin

SUSTech

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Motivation for Studying Toric Varieties

Overview

The study of toric varieties is a wonderful part of algebraic geometry.
There are elegant theorems and deep connections with polytopes, polyhedra, combinatorics, commutative algebra, and topology.

Fulton's Insight

As noted by Fulton, "toric varieties have provided a remarkably fertile testing ground for general theories."

Educational Value

The concreteness of toric varieties provides an excellent context for someone encountering the powerful techniques of modern algebraic geometry for the first time.

The Torus.

A *torus* $T \cong (\mathbb{C}^*)^n$ as an affine variety and a group.

Character

$$\chi : T \to \mathbb{C}^*$$

For any torus T, its characters form a free abelian group M of rank equal to $\dim T$ and $m \in M$ gives the character $\chi^m : T \to \mathbb{C}^*$.

For example, $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ gives a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$

$$\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \cdots t_n^{a_n}$$

All characters of $(\mathbb{C}^*)^n$ arise this way.
The characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$.

One-parameter subgroup

$$\lambda : \mathbb{C}^* \to T$$

For any T, the one-parameter subgroups form a free abelian group N of rank equal to the $\dim T$ and $u \in N$ gives the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T$.

For example, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$

$$\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n})$$

Consider $\langle \, , \, \rangle : M \times N \to \mathbb{Z}$:

- (Intrinsic) $\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ given by $t \mapsto t^\ell$ for some $\ell \in \mathbb{Z}$. Then $\langle m, u \rangle = \ell$.
- (Concrete) If $T = (\mathbb{C}^*)^n$ with $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$,

$$\langle m, u \rangle = \sum_{i=1}^{n} a_i b_i$$

The Definition of Affine Toric Variety.

Definition

An *affine toric variety* is an irreducible affine variety V containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on V.
(By algebraic action, we mean an action $T_N \times V \to V$ given by a morphism.)

Example

$C = \mathrm{V}(x^3 - y^2) \subseteq \mathbb{C}^2$
$C \setminus \{0\} = C \cap (\mathbb{C}^*)^2 = \{(t^2, t^3) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^* \quad t \mapsto (t^2, t^3)$

Example

$V = \mathrm{V}(xy - zw) \subseteq \mathbb{C}^4$
$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \mid t_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^3 \quad (t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$

A *lattice* is a free abelian group of finite rank.

A torus T_N has lattices M (of characters) and N (of one-parameter subgroups).
A set $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$ gives characters $\chi^{m_i} : T_N \to \mathbb{C}^*$.
Consider

$$\Phi_{\mathcal{A}} : T_N \longrightarrow \mathbb{C}^s$$

$$\Phi_{\mathcal{A}}(t) = (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \in \mathbb{C}^s.$$

$Y_{\mathcal{A}} := \overline{\operatorname{Im} \Phi_{\mathcal{A}}}$ is an affine toric variety.

Example

Let $\Phi_{\mathcal{A}} : (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^{d+1}$ defined by $(s,t) \mapsto (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)$ extend to

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^{d+1}$$

Thus $Y_{\mathcal{A}} = \operatorname{Im} \Phi =: \hat{C}_d$, which is called the *rational normal cone of degree* d.

Lemma

Let T_1 and T_2 be tori and let $\Phi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then the image of Φ is a torus and is closed in T_2.

The map $\Phi_{\mathcal{A}}$ can be regarded as a map of tori

$$\Phi_{\mathcal{A}} : T_N \longrightarrow (\mathbb{C}^*)^s.$$

The image $T = \Phi_{\mathcal{A}}(T_N)$ is a torus.
The torus T of $Y_{\mathcal{A}}$ has character lattice $\mathbb{Z}\mathcal{A}$.
The dimension of $Y_{\mathcal{A}}$ is the rank of $\mathbb{Z}\mathcal{A}$.

Toric Ideals.

$\Phi_{\mathcal{A}}$ induces

$$\hat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \longrightarrow M$$

that sends the standard basis $e_1, \ldots, e_s$ to $m_1, \ldots, m_s$.
Consider the exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow M.$$

Given $\ell = (\ell_1, \ldots, \ell_s) \in L$, set

$$\ell_+ = \sum_{\ell_i > 0} \ell_i e_i \quad \text{and} \quad \ell_- = -\sum_{\ell_i < 0} \ell_i e_i.$$

Note that $\ell = \ell_+ - \ell_-$ and the binomial

$$x^{\ell_+} - x^{\ell_-} = \prod_{\ell_i > 0} x_i^{\ell_i} - \prod_{\ell_i < 0} x_i^{-\ell_i}$$

vanishes on $Y_{\mathcal{A}}$.

$$\mathrm{I}(Y_{\mathcal{A}}) = \langle x^{\ell_+} - x^{\ell_-} \mid \ell \in L \rangle = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \rangle.$$

Definition

Let $L \subseteq \mathbb{Z}^s$ be a sublattice.

- The ideal $I_L := \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s$ and $\alpha - \beta \in L\rangle$ is called a *lattice ideal*.
- A prime lattice ideal is called a *toric ideal*.

Example

- $\langle x^3 - y^2 \rangle \subseteq \mathbb{C}[x, y]$
- $\langle xz - yw \rangle \subseteq \mathbb{C}[x, y, z, w]$
- $\langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1 \rangle \subseteq \mathbb{C}[x_0, \ldots, x_d]$
  The ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix}$$

  which is the ideal of the rational normal cone $\hat{C}_d$

Affine Semigroups.

Definition (Affine semigroup)

- A *semigroup* is a set S with an associative binary operation and an identity element.
- A *affine semigroup* is a finitely generated commutative semigroup S, which can be embedded in a lattice M.
  - There exists a finite set $\mathcal{A} \subseteq M$ such that

$$\mathbb{N}\mathcal{A} = \left\{ \sum_{m \in \mathcal{A}} a_m m \mid a_m \in \mathbb{N} \right\} = S.$$

- A *semigroup algebra* $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication induced by the semigroup structure of S.

The simplest example of an affine semigroup is $\mathbb{N}^n \subseteq \mathbb{Z}^n$.
Given a lattice M and a finite set $\mathcal{A} \subseteq M$, we get the affine semigroup $\mathbb{N}\mathcal{A} \subseteq M$.
Up to isomorphism, all affine semigroups are of this form.

We think of M as the character lattice of a torus T_N.

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m \mid c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m \right\},$$

with multiplication induced by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

If $S = \mathbb{N}\mathcal{A}$ for $\mathcal{A} = \{m_1, \ldots, m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$.

- The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

  $$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n],$$

  where $x_i = \chi^{e_i}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$.
- If $e_1, \ldots, e_n$ is a basis of a lattice M, then M is generated by $\mathcal{A} = \{\pm e_1, \ldots, \pm e_n\}$ as an affine semigroup. Setting $t_i = \chi^{e_i}$ gives

  $$\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

  $\mathbb{C}[M]$ is the coordinate ring of the torus T_N.

Let $S \subseteq M$ be an affine semigroup. Then $\mathrm{Spec}(\mathbb{C}[S])$ is an affine toric variety whose torus has character lattice $\mathbb{Z}S$, and if $S = \mathbb{N}\mathcal{A}$ for a finite set $\mathcal{A} \subseteq M$, then $\mathrm{Spec}(\mathbb{C}[S]) = Y_{\mathcal{A}}$.

Equivalence of Constructions.

Let V be an affine variety. TFAE:

- V is an affine toric variety .
- $V = Y_{\mathcal{A}}$ for a finite set $\mathcal{A}$ in a lattice.
- V is an affine variety defined by a toric ideal.
- $V = \mathrm{Spec}(\mathbb{C}[S])$ for an affine semigroup S.

Example

$C = \mathrm{V}(x^3 - y^2) \subseteq \mathbb{C}^2$

$T_N = C \setminus \{0\} = C \cap (\mathbb{C}^*)^2 = \{(t^2, t^3) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^* \qquad t \mapsto (t^2, t^3)$

$\mathcal{A} = \{2, 3\} \qquad \Phi_{\mathcal{A}}(t) = (t^2, t^3)$

$Y_{\mathcal{A}}$ is the curve $x^3 = y^2$. $\qquad \mathrm{I}(Y_{\mathcal{A}}) = \langle x^3 - y^2 \rangle$

$S = \{0, 2, 3, \ldots\} \qquad \mathbb{C}[S] = \mathbb{C}[t^2, t^3] \simeq \mathbb{C}[x, y]/\langle x^3 - y^2 \rangle$

Example

$V = \mathrm{V}(xy - zw) \subseteq \mathbb{C}^4$
$T_N \simeq (\mathbb{C}^*)^3$
$(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$
The lattice points used in this map can be represented as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$\mathrm{I}(Y_{\mathcal{A}}) = \langle xy - zw \rangle \subseteq \mathbb{C}[x, y, z, w]$
The corresponding semigroup $S \subseteq \mathbb{Z}^3$ consists of the $\mathbb{N}$-linear combinations of the column vectors.

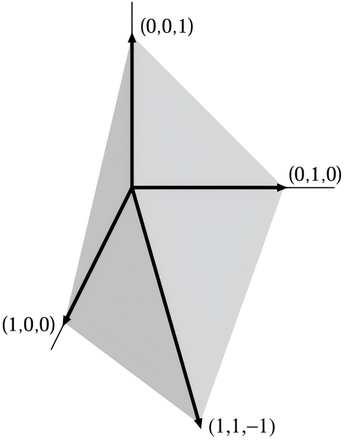
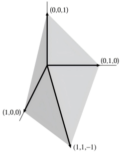


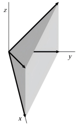
Figure: Cone corresponding to $V = \mathrm{V}(xy - zw)$

Convex Polyhedral Cones.

Fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$.

- A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form
$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}},$$
where $S \subseteq N_{\mathbb{R}}$ is finite.
- Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its *dual cone* is defined by
$$\sigma^{\vee} = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \}.$$





A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is *rational* if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$. The ray generators of the edges is the *minimal generators* of $\sigma = \mathrm{Cone}(S)$.

Cone and Semigroup Algebras.

Given a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, $S_\sigma = \sigma^\vee \cap M$ is an affine semigroup. Then

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety.

Example

Fix $0 \leq r \leq n$ and set $\sigma = \mathrm{Cone}(e_1, \ldots, e_r) \subseteq \mathbb{R}^n$. Then

$$\sigma^\vee = \mathrm{Cone}(e_1, \ldots, e_r, \pm e_{r+1}, \ldots, \pm e_n)$$
$$U_\sigma = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}])$$

Faces

- Given $m \neq 0$ in $M_{\mathbb{R}}$,

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}}$$
$$H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}$$

  If $\sigma \subseteq H_m^+$, then H_m is a supporting hyperplane and H_m^+ is a supporting half-space.

$$\sigma^\vee = \text{Cone}(m_1, \ldots, m_s) \longleftrightarrow \sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+.$$

- A *face* of σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$, written $\tau \preceq \sigma$.
  - Using $m = 0$, $\sigma \preceq \sigma$.
  - Every face of σ is a polyhedral cone.
  - An intersection of two faces of σ is again a face of σ.
  - A face of a face of σ is again a face of σ.
- A *facet* of σ is a face τ of codimension 1, i.e., $\dim \tau = \dim \sigma - 1$.
- An *edge* of σ is a face of dimension 1.

- Given a face $\tau \preceq \sigma \subseteq N_{\mathbb{R}}$, define

$$\begin{aligned}
\tau^{\perp} &= \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\} \\
\tau^{*} &= \{m \in \sigma^{\vee} \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\} \\
&= \sigma^{\vee} \cap \tau^{\perp}.
\end{aligned}$$

  We call τ^* the *dual face* of τ.
- A cone is called *strongly convex* if the origin is a face.

$$\begin{aligned}
\sigma \text{ is strongly convex} &\iff \{0\} \text{ is a face of } \sigma \\
&\iff \sigma \text{ contains no positive-dimensional subspace of } N_{\mathbb{R}} \\
&\iff \sigma \cap (-\sigma) = \{0\} \\
&\iff \dim \sigma^{\vee} = n.
\end{aligned}$$

- Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.
  - σ is *smooth* or *regular* if its minimal generators form part of a $\mathbb{Z}$-basis of N
  - σ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.

Faces and Affine Open Subsets.

Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and let $\tau \preceq \sigma$ be a face.
Then we can find $m \in \sigma^\vee \cap M$ such that $\tau = H_m \cap \sigma$.
Then $\mathbb{C}[S_\tau] = \mathbb{C}[\tau^\vee \cap M]$ is the localization of $\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ at $\chi^m \in \mathbb{C}[S_\sigma]$.
In other words,

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}.$$

$$U_\tau = \mathrm{Spec}(\mathbb{C}[S_\tau]) = \mathrm{Spec}(\mathbb{C}[S_\sigma]_{\chi^m}) = (\mathrm{Spec}(\mathbb{C}[S_\sigma]))_{\chi^m} \subseteq U_\sigma.$$

If $\tau = \sigma_1 \cap \sigma_2$, then

$$\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m,$$

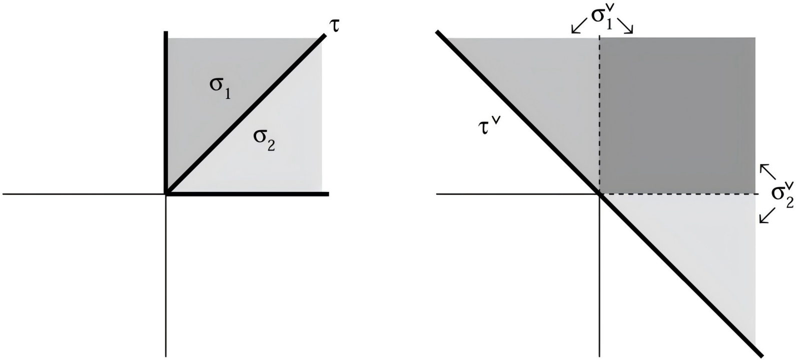
for some $m \in \sigma_1^\vee \cap (-\sigma_2^\vee) \cap M$.
This shows that

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}.$$

- A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that:
  - Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
  - For all $\sigma \in \Sigma$, each face of σ is also in Σ.
  - For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence also in Σ).

Example

Let $\sigma_1 = \text{Cone}(e_1 + e_2, e_2)$, and let $\sigma_2 = \text{Cone}(e_1, e_1 + e_2)$ in $N_{\mathbb{R}} = \mathbb{R}^2$.
Then $\tau = \sigma_1 \cap \sigma_2 = \text{Cone}(e_1 + e_2)$.
The dual cones $\sigma_1^\vee = \text{Cone}(e_1, -e_1 + e_2)$, $\sigma_2^\vee = \text{Cone}(e_1 - e_2, e_2)$, and $\tau^\vee = \sigma_1^\vee + \sigma_2^\vee$.
Note $\tau = \sigma_1 \cap H_m = \sigma_2 \cap H_{-m}$, where $m = -e_1 + e_2 \in \sigma_1^\vee$ and $-m = e_1 - e_2 \in \sigma_2^\vee$.



The Toric Variety of a Fan.

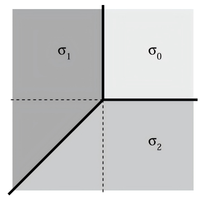
Consider the collection of $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$, where σ runs over all cones in a fan Σ. Let σ_1 and σ_2 be any two of these cones and let $\tau = \sigma_1 \cap \sigma_2$. We have an isomorphism

$$g_{\sigma_2,\sigma_1} : (U_{\sigma_1})_{\chi^m} \simeq (U_{\sigma_2})_{\chi^{-m}}$$

which is the identity on U_τ.
The compatibility conditions for gluing the affine varieties U_σ along the subvarieties $(U_\sigma)_{\chi^m}$ are satisfied.
Hence we obtain an abstract variety X_Σ associated to the fan Σ.

Example

The fan Σ has three 2-dimensional cones
$\sigma_0 = \mathrm{Cone}(e_1, e_2)$,
$\sigma_1 = \mathrm{Cone}(-e_1 - e_2, e_2)$,
$\sigma_2 = \mathrm{Cone}(e_1, -e_1 - e_2)$,
together with the three rays
$\tau_{ij} = \sigma_i \cap \sigma_j$ for $i \neq j$,
and the origin.



The toric variety X_Σ is covered by the affine opens

$$\begin{aligned} U_{\sigma_0} &= \mathsf{Spec}(\mathbb{C}[S_{\sigma_0}]) \simeq \mathsf{Spec}(\mathbb{C}[x,y]) \\ U_{\sigma_1} &= \mathsf{Spec}(\mathbb{C}[S_{\sigma_1}]) \simeq \mathsf{Spec}(\mathbb{C}[x^{-1},x^{-1}y]) \\ U_{\sigma_2} &= \mathsf{Spec}(\mathbb{C}[S_{\sigma_2}]) \simeq \mathsf{Spec}(\mathbb{C}[xy^{-1},y^{-1}]). \end{aligned}$$

The gluing data on the coordinate rings is given by

$$\begin{aligned} g_{10}^* &: \mathbb{C}[x,y]_x \simeq \mathbb{C}[x^{-1},x^{-1}y]_{x^{-1}} \\ g_{20}^* &: \mathbb{C}[x,y]_y \simeq \mathbb{C}[xy^{-1},y^{-1}]_{y^{-1}} \\ g_{21}^* &: \mathbb{C}[x^{-1},x^{-1}y]_{x^{-1}y} \simeq \mathbb{C}[xy^{-1},y^{-1}]_{xy^{-1}}. \end{aligned}$$

If we use the usual homogeneous coordinates (x_0,x_1,x_2) on $\mathbb{P}^2$, then $x \mapsto \frac{x_1}{x_0}$ and $y \mapsto \frac{x_2}{x_0}$ identifies the standard affine open $U_i \subseteq \mathbb{P}^2$ with $U_{\sigma_i} \subseteq X_\Sigma$. Hence we have recovered $\mathbb{P}^2$ as X_Σ.

If the fan Σ has only one 2-dimensional cone $\sigma = \mathbb{R}^2$, note that in this case σ is not strongly convex, $\sigma^\vee = \text{Cone}(e_1)$, then

$$U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = \text{Spec}(\mathbb{C}).$$

If the fan Σ has only one 2-dimensional cone $\sigma = \text{Cone}(e_1, e_2, -e_2)$, note that in this case Σ is also not strongly convex, $\sigma^\vee = \text{Cone}(e_1)$, then

$$U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = \text{Spec}(\mathbb{C}[x_1]).$$

In general,

$$\dim U_\sigma = n \iff \text{ the torus of } U_\sigma \text{ is } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \iff \sigma \text{ is strongly convex.}$$

This is why we emphasize the condition of strongly convexity.

Polytopes.

A *polytope* in $M_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \left\{ \sum_{m \in S} \lambda_m m \mid \lambda_m \geq 0, \sum_{m \in S} \lambda_m = 1 \right\} \subseteq M_{\mathbb{R}},$$

Given a nonzero vector u in the dual space $N_{\mathbb{R}}$ and $b \in \mathbb{R}$, define

$$H_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = b\} \quad \text{and} \quad H^+_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq b\}.$$

A subset $Q \subseteq P$ is a *face* of P, written $Q \preceq P$, if there are $u \in N_{\mathbb{R}} \setminus \{0\}, b \in \mathbb{R}$ with

$$Q = H_{u,b} \cap P \quad \text{and} \quad P \subseteq H^+_{u,b}.$$

$$P = \bigcap_{F \text{ facet}} H^+_F = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P\}$$

where $(u_F, a_F) \in N_{\mathbb{R}} \times \mathbb{R}$ is unique up to multiplication by a positive real number.

Application

Let X be a projective simplicial toric variety of dimension n, associated with some complete fan in a lattice $N \cong \mathbb{Z}^n$. The set of rays of Σ is denoted by $\Sigma(1)$. There exists a bijection between prime torus-invariant divisors and rays. The divisors are symbolized by $\{D_i \mid i \in \Sigma(1)\}$. Let $I \subset \Sigma(1)$ and $J = \Sigma(1) - I$. Denote

$$O_J = \{D \sim \sum d_\rho D_\rho \mid d_\rho > 0, \rho \in I \text{ and } d_\rho < 0, \rho \in J\}.$$

Let P_X^J denote the subset of P_X consisting of the union of all facets corresponding to rays in J.

$$\overline{\mathrm{Amp}_q(X)} = \mathrm{Pic}(X) - \bigcup_{i \geq q} \bigcup_{\widetilde{H}^i(P_X^J) \neq 0} [O_J].$$