Thom spectra, infinite loop spaces, generalized cocycles and σ-orientation

Jiacheng Liang

SUSTech Department of Mathematics

May 23, 2023

Table of Contents

Thom spectrum functor

Definition (Thom spectrum functor)

Let $(f : X \to BO) \in Top_{\downarrow BO}$, then the standard filtration $X_V = f^{-1}(BO(V))$ gives a Thom prespectrum

$$M_p(f)(V) = Th(E(X_V) \to X_V) = E(X_V)_+ \wedge_{O(V)_+} S^V$$

The spectrification $M(f)$ of $M_p(f)$ is called the Thom spectrum corresponding f.

Remark

(i) Actually, any filtration $\varinjlim_{V \subset \mathbb{R}^\infty} F_V X = X$ where $F_V X$ is a closed subspace of X such that $F_V X \subset X_V$ gives the same Thom spectra (though not the same prespectra).
(ii) For $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$, almost all arguments about Thom spectra throughout this talk apply for them. In the following content we always use $O(\infty)$ for convenience.

Thom spectrum functor

Definition (Thom spectrum functor)

Let $(f : X \to BO) \in Top_{\downarrow BO}$, then the standard filtration $X_V = f^{-1}(BO(V))$ gives a Thom prespectrum

$$M_p(f)(V) = Th(E(X_V) \to X_V) = E(X_V)_+ \wedge_{O(V)_+} S^V$$

The spectrification $M(f)$ of $M_p(f)$ is called the Thom spectrum corresponding f.

Remark

(i) Actually, any filtration $\varinjlim_{V \subset \mathbb{R}^\infty} F_V X = X$ where $F_V X$ is a closed subspace of X such that $F_V X \subset X_V$ gives the same Thom spectra (though not the same prespectra).
(ii) For $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$, almost all arguments about Thom spectra throughout this talk apply for them. In the following content we always use $O(\infty)$ for convenience.

Properties of the Thom spectrum functor

For any spectrum $E \in Sp$ and any $V \subset \mathbb{R}^\infty$, $\Omega^\infty E$ admits a right $O(V)$-action since $\Omega^\infty E = E_0 = \Omega^V E_V = F(S^V, E_V)$. These actions are coherent between different V, so we actually get a right O-action on $\Omega^\infty E$.

Theorem

The Thom spectrum functor induces a continuous adjoint pair

$$Top_{\downarrow BO} \underset{EO\times_O \Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$$

Given a map $(f : X \to BO) \in \mathcal{U}/BO$ and $E \in Sp$, then

$$\mathrm{Hom}_{Sp}(Mf, E) = \mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BO}(X, EO \times_O \Omega^\infty E)$$

Properties of the Thom spectrum functor

For any spectrum $E \in Sp$ and any $V \subset \mathbb{R}^\infty$, $\Omega^\infty E$ admits a right $O(V)$-action since $\Omega^\infty E = E_0 = \Omega^V E_V = F(S^V, E_V)$. These actions are coherent between different V, so we actually get a right O-action on $\Omega^\infty E$.

Theorem

The Thom spectrum functor induces a continuous adjoint pair

$$Top_{\downarrow BO} \underset{EO \times_O \Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$$

Given a map $(f : X \to BO) \in \mathcal{U}/BO$ *and* $E \in Sp$, *then*

$$\mathrm{Hom}_{Sp}(Mf, E) = \mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BO}(X, EO \times_O \Omega^\infty E)$$

Proof

First

$$\mathrm{Hom}_{Sp}(Mf, E) = \mathrm{Hom}_{Sp}(\mathrm{colim}_V MX_V, E) = \lim_V \mathrm{Hom}_{Sp}(MX_V, E)$$

Second we define EX_V and $Z(V)$ by pullback diagrams,

$$\begin{array}{ccc} EX_V & \longrightarrow & EO(V) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & BO(V) \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_V & \longrightarrow & EO(V) \times_{O(V)} O \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & BO(V) \end{array}$$

then

$$\lim_V \mathrm{Hom}_{Sp}(MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{U}_*}(EX_{V+} \wedge_{O(V)} S^V, E_V) =$$
$$\lim_V \mathrm{Hom}_{\mathcal{U}_*[O_{V+}]}(EX_{V+}, \Omega^V E_V) = \lim_V \mathrm{Hom}_{\mathcal{U}[O_V]}(EX_V, \Omega^\infty E) =$$

$$\lim_V \mathrm{Hom}_{\mathcal{U}[O]}(EX_V \times_{O(V)} O, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[O]}(Z_V, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E)$$

Since equivariant maps from a principle G-bundle to a G-space are equivalent to sections of the associated bundle, i.e.

$$\mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/X}(X, f^* EO \times_O \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BO}(X, EO \times_O \Omega^\infty E)$$

Proof

First

$$\mathrm{Hom}_{\mathcal{S}p}(Mf, E) = \mathrm{Hom}_{\mathcal{S}p}(\mathrm{colim}_V MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{S}p}(MX_V, E)$$

Second we define EX_V and $Z(V)$ by pullback diagrams,

$$\begin{array}{ccc} EX_V & \longrightarrow & EO(V) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & BO(V) \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_V & \longrightarrow & EO(V) \times_{O(V)} O \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & BO(V) \end{array}$$

then

$$\lim_V \mathrm{Hom}_{\mathcal{S}p}(MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{U}_*}(EX_{V+} \wedge_{O(V)} S^V, E_V) =$$
$$\lim_V \mathrm{Hom}_{\mathcal{U}_*[O_{V+}]}(EX_{V+}, \Omega^V E_V) = \lim_V \mathrm{Hom}_{\mathcal{U}[O_V]}(EX_V, \Omega^\infty E) =$$

$$\lim_V \mathrm{Hom}_{\mathcal{U}[O]}(EX_V \times_{O(V)} O, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[O]}(Z_V, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E)$$

Since equivariant maps from a principle G-bundle to a G-space are equivalent to sections of the associated bundle, i.e.

$$\mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/X}(X, f^* EO \times_O \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BO}(X, EO \times_O \Omega^\infty E)$$

Proof

First

$$\mathrm{Hom}_{\mathcal{S}p}(Mf, E) = \mathrm{Hom}_{\mathcal{S}p}(\mathrm{colim}_V MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{S}p}(MX_V, E)$$

Second we define EX_V and $Z(V)$ by pullback diagrams,

$$\begin{array}{ccc} EX_V & \longrightarrow & EO(V) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & BO(V) \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_V & \longrightarrow & EO(V) \times_{O(V)} O \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & BO(V) \end{array}$$

then

$$\lim_V \mathrm{Hom}_{\mathcal{S}p}(MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{U}_*}(EX_{V+} \wedge_{O(V)} S^V, E_V) =$$
$$\lim_V \mathrm{Hom}_{\mathcal{U}_*[O_{V+}]}(EX_{V+}, \Omega^V E_V) = \lim_V \mathrm{Hom}_{\mathcal{U}[O_V]}(EX_V, \Omega^\infty E) =$$

$$\lim_V \mathrm{Hom}_{\mathcal{U}[O]}(EX_V \times_{O(V)} O, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[O]}(Z_V, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E)$$

Since equivariant maps from a principle G-bundle to a G-space are equivalent to sections of the associated bundle, i.e.

$$\mathrm{Hom}_{\mathcal{U}[O]}(f^* EO, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/X}(X, f^* EO \times_O \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BO}(X, EO \times_O \Omega^\infty E)$$

Properties of the Thom spectrum functor

Proposition

This adjunction $Top_{\downarrow BO} \underset{EO\times_O \Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$ *is actually a Quillen adjunction since* $M(S^{n-1} \to D^n)$ *is a cell pair of spectra and* $M(D^n \times 0 \to D^n \times I)$ *is a weak equivalent cell pair for those morphisms over* BO.

Proposition

Let $f : X \to BO$ *be a map and* A *a space. Let* g *be the composite* $X \times A \to X \to BO$, *where the first map is the projection away from* A. *Then* $T(g) = A_+ \wedge T(f)$, *which implies Thom spectrum functor preserves tensors, and hence is a topological Quillen functor.*

Proposition

Thom spectrum functor $T(-)$ *preserves weak equivalences. Any Thom spectrum* $T(f)$ *from a map* $F : X \to BO$ *is* (-1)*-connective.*

Properties of the Thom spectrum functor

Proposition

This adjunction $Top_{\downarrow BO} \underset{EO\times_O\Omega^\infty(-)}{\overset{M(-)}{\rightleftarrows}} Sp$ *is actually a Quillen adjunction since* $M(S^{n-1} \to D^n)$ *is a cell pair of spectra and* $M(D^n \times 0 \to D^n \times I)$ *is a weak equivalent cell pair for those morphisms over* BO.

Proposition

Let $f : X \to BO$ *be a map and* A *a space. Let* g *be the composite* $X \times A \to X \to BO$, *where the first map is the projection away from* A. *Then* $T(g) = A_+ \wedge T(f)$, *which implies Thom spectrum functor preserves tensors, and hence is a topological Quillen functor.*

Proposition

Thom spectrum functor $T(-)$ *preserves weak equivalences. Any Thom spectrum* $T(f)$ *from a map* $F : X \to BO$ *is* (-1)*-connective.*

Proposition

Let $\mathcal{V}_1, \mathcal{V}_2$ be two real universes.
(i) Given maps $B \to \mathcal{L}(V_1, V_2)$ and $f : X \to BO(\mathcal{V}_1)$, denote g to be the composition $B \times X \to B \times BO(\mathcal{V}_1) \to BO(\mathcal{V}_2)$. Then we have the natural isomorphism $T(g) \cong B \ltimes T(f)$.
(ii) Given maps $f : X \to BO(\mathcal{V}_1)$ and $g : Y \to BO(\mathcal{V}_2)$, denote $f \times g$ to be the composition $X \times Y \to BO(\mathcal{V}_1) \times BO(\mathcal{V}_2) \to BO(\mathcal{V}_1 \oplus \mathcal{V}_2)$. Then $T(f \times g) \cong T(f) \,\bar{\wedge}\, T(g)$.

Proposition

Let $\mathcal{L}(n) = \mathcal{L}(\mathbb{R}^{\infty \times n}, \mathbb{R}^{\infty})$, then for any map $f : X \to BO$ we have

$$T(\coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} X^n \to \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} BO^n \to BO) = \bigvee_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} T(f)^{\bar{\wedge} n}$$

Monad and Thom spectrum functor

Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be topological bicomplete categories, and $\mathbb{A} : \mathcal{C} \to \mathcal{C}$ and $\mathbb{B} : \mathcal{D} \to \mathcal{D}$ be continuous monads. Further suppose that there is a continuous functor $F : \mathcal{C} \to \mathcal{D}$ which is coherent with the monad structure and therefore yields a functor $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$.
If $F : \mathcal{C} \to \mathcal{D}$ is left adjoint functor preserving tensors, and the monads $\mathbb{A}$ and $\mathbb{B}$ preserve reflexive coequalizers, then $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$ is still a left adjoint functor preserving tensors.

Theorem

Thom spectrum functor induces topological Quillen adjoint pairs

$$Top[\mathcal{L}(1)]_{\downarrow BO} \rightleftharpoons Sp[\mathcal{L}(1)] \quad \text{and} \quad Top[E_\infty]_{\downarrow BO} \rightleftharpoons Sp[E_\infty].$$

Diagonal and Thom isomorphism

Definition (coaction)

For any map $f : X \to BO$, the diagonal induces a coaction $X \to X \times X$ in $Top_{\downarrow BO}$, where $X \times X \to BO$ is the projection of the second variable. It gives a natural coaction on Thom spectra: $Mf \to X_+ \wedge Mf$.

Definition (Thom morphism)

With the same hypothesis above, given a homotopy commutative ring spectrum E and a morphism of spectra $Mf \to E$ we have a natural morphism $E \wedge Mf \to E \wedge X_+ \wedge Mf \to E \wedge X_+ \wedge E \to E \wedge X_+$ in $Ho(Sp)$. It induces a natural homological morphism $\phi_f : E_*(Mf) \to E_*(X)$.

Under certain condition ϕ_f will be an isomorphism, which is called Thom isomorphism.

Diagonal and Thom isomorphism

Theorem (Thom isomorphism)

Let $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$ or $Spin(\infty)$. Let E be a homotopy commutative ring (phantom-)spectrum.
(i) Given a (phantom) ring spectrum morphism $MG \to E$, then for any map $X \to BG$ the Thom morphism $E_*(Mf) \to E_*(X)$ is an isomorphism.
Moreover, if X is E_∞ and f is an E_∞ map, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.
(ii) Given an E_∞ space X and an E_∞ map $f : X \to BG$. Let $Mf \to E$ be a (phantom) ring spectrum morphism. If X is 0-connected, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.

Example

Let $MO \to H\mathbb{Z}/2$ and $MU \to H\mathbb{Z}$ be ring spectrum morphisms from the 0-th postnikov tower. Then we have natural Thom isomorphisms $H_*(MO;\mathbb{Z}/2) \to H_*(BO;\mathbb{Z}/2)$ and $H_*(MU) \to H_*(BU)$.

Diagonal and Thom isomorphism

Theorem (Thom isomorphism)

Let $G = Sp(\infty), U(\infty), SU(\infty), O(\infty), SO(\infty)$ or $Spin(\infty)$. Let E be a homotopy commutative ring (phantom-)spectrum.
(i) Given a (phantom) ring spectrum morphism $MG \to E$, then for any map $X \to BG$ the Thom morphism $E_*(Mf) \to E_*(X)$ is an isomorphism.
Moreover, if X is E_∞ and f is an E_∞ map, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.
(ii) Given an E_∞ space X and an E_∞ map $f : X \to BG$. Let $Mf \to E$ be a (phantom) ring spectrum morphism. If X is 0-connected, then $E_*(Mf) \to E_*(X)$ is an isomorphism of E_*-algebras.

Example

Let $MO \to H\mathbb{Z}/2$ and $MU \to H\mathbb{Z}$ be ring spectrum morphisms from the 0-th postnikov tower. Then we have natural Thom isomorphisms $H_*(MO;\mathbb{Z}/2) \to H_*(BO;\mathbb{Z}/2)$ and $H_*(MU) \to H_*(BU)$.

Infinite loop space machine

We first introduce some consequences of infinite loop space machine.

Theorem (Additive infinite loop space machine, in ABGHR)

Let C be a cofibrant unital E_∞ operad in Top and $f : C_* \to \Omega^\infty\Sigma^\infty$ be a morphism of monads on Top_*. Then the Quillen pair (Σ^f, Ω^f) induces a equivalence of categories enriched in $Ho(Top)$ if we restrict it to the following subcategories

$$\text{group-like } Ho(E_\infty\text{-spaces}) \rightleftharpoons (-1)\text{-connective } Ho(Sp)$$

where $\Sigma^f(-) = \Sigma^\infty \otimes_{C_*} (-)$ is the coequalizer of the following diagram in Sp

$$\begin{array}{ccccc} \Sigma^\infty C_* X & \overset{\Sigma^\infty \mu}{\rightrightarrows} & \Sigma X & \longrightarrow & \Sigma^f X \\ & \searrow & & \nearrow & \\ & & \Sigma^\infty\Omega^\infty\Sigma^\infty X & & \end{array}$$

And $\Omega^f X = \Omega^\infty X$ is endowed with the C_*-action $C_*\Omega^\infty X \to \Omega^\infty\Sigma^\infty\Omega^\infty X \to \Omega^\infty X$.

Uniqueness of infinite loop space machine

Theorem (Uniqueness of additive infinite loop space machine, May 77)

We define an infinite loop space machine to be an adjoint pair (F, G)

$Ho(E_\infty\text{-spaces}) \underset{G}{\overset{F}{\rightleftarrows}} (-1)\text{-connective } Ho(Sp)$ *such that*

(1) The composition

$(-1)\text{-connective } Ho(Sp) \xrightarrow{G} Ho(E_\infty\text{-spaces}) \to CMon(Ho(\mathrm{Top}_*))$ *is equivalent to* Ω^∞;

(2) For any $X \in Ho(E_\infty\text{-spaces})$, $X \to GF(X)$ *is a group completion, which means* $\pi_0 GF(X)$ *is a group and* $H_*(X)[(\pi_0 X)^{-1}] \to H_* GF(X)$ *is isomorphic.*

Now, if (F_1, G_1) *and* (F_2, G_2) *are two infinite loop space machines, then there exists a natural equivalence between* F_1 *and* F_2.

Remark

The existence of infinite loop space implies that for any group-like E_∞-space X, the induced pointed H-space of it is actually an H-group because $X \cong \Omega^\infty FX$ in $CMon(Ho(\mathrm{Top}_*))$ and $\Omega^\infty FX$ is a pointed H-group.

Uniqueness of infinite loop space machine

Theorem (Uniqueness of additive infinite loop space machine, May 77)

We define an infinite loop space machine to be an adjoint pair (F, G)

$$Ho(E_\infty\text{-spaces}) \underset{G}{\overset{F}{\rightleftarrows}} (-1)\text{-connective } Ho(Sp)$$ *such that*

(1) The composition

(-1)*-connective* $Ho(Sp) \xrightarrow{G} Ho(E_\infty\text{-spaces}) \to CMon(Ho(\mathsf{Top}_*))$ *is equivalent to* Ω^∞;

(2) For any $X \in Ho(E_\infty\text{-spaces})$, $X \to GF(X)$ *is a group completion, which means* $\pi_0 GF(X)$ *is a group and* $H_*(X)[(\pi_0 X)^{-1}] \to H_* GF(X)$ *is isomorphic.*

Now, if (F_1, G_1) *and* (F_2, G_2) *are two infinite loop space machines, then there exists a natural equivalence between* F_1 *and* F_2.

Remark

The existence of infinite loop space implies that for any group-like E_∞-space X, the induced pointed H-space of it is actually an H-group because $X \cong \Omega^\infty FX$ in $CMon(Ho(\mathsf{Top}_*))$ and $\Omega^\infty FX$ is a pointed H-group.

We also consider the connective complex K-theory bu, so $bu^* = \mathbb{Z}[v], |v| = -2$ and $bu^{2t}(X) = [X, BU\langle 2t\rangle]$. To make this true when $t = 0$, we adopt the convention that $BU\langle 0\rangle = \mathbb{Z} \times BU$. Multiplication by $v^t : \Sigma^{2t} bu \to bu$ gives the $(2t-1)$-connective cover of bu. We define $BU\langle 2t\rangle = \Omega^\infty \Sigma^{2t} bu$. Under this identification, we a sequence of morphisms in $Ho(Top[E_\infty])$

$$... \to BU\langle 2k\rangle \to ... \to BU\langle 6\rangle \to BSU \to BU \to BU\langle 0\rangle$$

derived from infinite loop space machine.

Proposition

Combine with the Thom spectrum functor $Top[E_\infty]_{\downarrow BU} \rightleftharpoons Sp[E_\infty]$, the sequence above induces a new sequence of morphisms in $Ho(Sp[E_\infty])$

$$... \to MU\langle 2k\rangle \to ... \to MU\langle 6\rangle \to MSU \to MU \to MP.$$

We also consider the connective complex K-theory bu, so $bu^* = \mathbb{Z}[v], |v| = -2$ and $bu^{2t}(X) = [X, BU\langle 2t\rangle]$. To make this true when $t = 0$, we adopt the convention that $BU\langle 0\rangle = \mathbb{Z} \times BU$. Multiplication by $v^t : \Sigma^{2t} bu \to bu$ gives the $(2t-1)$-connective cover of bu. We define $BU\langle 2t\rangle = \Omega^\infty \Sigma^{2t} bu$. Under this identification, we a sequence of morphisms in $Ho(Top[E_\infty])$

$$... \to BU\langle 2k\rangle \to ... \to BU\langle 6\rangle \to BSU \to BU \to BU\langle 0\rangle$$

derived from infinite loop space machine.

Proposition

Combine with the Thom spectrum functor $Top[E_\infty]_{\downarrow BU} \rightleftharpoons Sp[E_\infty]$, the sequence above induces a new sequence of morphisms in $Ho(Sp[E_\infty])$

$$... \to MU\langle 2k\rangle \to ... \to MU\langle 6\rangle \to MSU \to MU \to MP.$$

Firstly define the map $\rho_0 : P \to 1 \times BU \subset BU\langle 0\rangle$ just to be the map classifying the tautological line bundle L.
As for $t > 0$, let $L_1, \ldots, L_t$ be the obvious line bundles over P^t. Let $x_i \in bu^2\left(P^t\right)$ be the bu-theory Euler class, given by the formula

$$vx_i = 1 - L_i.$$

Then we have the isomorphisms

$$bu^*\left(P^t\right) \cong \mathbb{Z}[v][[x_1, \ldots, x_t]]$$

The class $\prod_i x_i \in bu^{2t}\left(P^t\right)$ gives the map $\rho_t : P^t = \left(\mathbb{C}P^\infty\right)^t \to BU\langle 2t\rangle$.

Remark

Note that the composition $P^t \xrightarrow{\rho_t} BU\langle 2t\rangle \to BU\langle 0\rangle$ classifies the bundle $\prod_i (1 - L_i)$.

Firstly define the map $\rho_0 : P \to 1 \times BU \subset BU\langle 0\rangle$ just to be the map classifying the tautological line bundle L.
As for $t > 0$, let $L_1, \ldots, L_t$ be the obvious line bundles over P^t. Let $x_i \in bu^2\left(P^t\right)$ be the bu-theory Euler class, given by the formula

$$vx_i = 1 - L_i.$$

Then we have the isomorphisms

$$bu^*\left(P^t\right) \cong \mathbb{Z}[v][[x_1, \ldots, x_t]]$$

The class $\prod_i x_i \in bu^{2t}\left(P^t\right)$ gives the map $\rho_t : P^t = \left(\mathbb{C}P^\infty\right)^t \to BU\langle 2t\rangle$.

Remark

Note that the composition $P^t \xrightarrow{\rho_t} BU\langle 2t\rangle \to BU\langle 0\rangle$ classifies the bundle $\prod_i (1 - L_i)$.

n-cocycles in a Cartesian monoidal category

Definition

Let C be a category admitting finite products. If A and T are abelian monoid objects in $CMon(C)$, we define $C^0(A, T)$ to be the group

$$C^0(A, T) \stackrel{\text{def}}{=} \mathrm{Hom}_C(A, T)$$

and for $k \geq 1$ we let $C^k(A, T)$ be the subgroup of $f \in \mathrm{Hom}_C(A^k, T)$ such that
(a) $f\,(a_1, \ldots, a_{k-1}, 0) = 0$
(b) $f\,(a_1, \ldots, a_k)$ is symmetric in the a_i;
(c) $f\,(a_1, a_2, a_3, \ldots, a_k) + f\,(a_0, a_1 + a_2, a_3, \ldots, a_k) =$
$f\,(a_0 + a_1, a_2, a_3, \ldots, a_k) + f\,(a_0, a_1, a_3, \ldots, a_k)$, when k ≥ 2.

Remark

We refer to (c) as the "cocycle" condition for f. If T is an abelian group object, then in definition (a) can be replaced by (a)': $f(0, 0, ..., 0) = 0$.

n-cocycles in algebraic geometry

From definition we can make n-cocycles a sheaf as the following: let X, Y are commutative momoid fppf sheaves over S, we define $\underline{C}^k(X, Y)(T) = C^k(X_T, Y_T)$. It is actually a representable commutative monoid sheaf in $Sh(Sch/S)_{fppf}$ in certain case.

Proposition

Let G be a formal group over a scheme S. Then for all k, the functor $\underline{C}^k(G, \mathbb{G}_m)$ is an S-affine commutative group scheme.

Proof: It suffices to work $k \geq 1$ and locally on S, so we may assume $S = \operatorname{Spec}(R)$ and choose a coordinate x on G. We define power series $g_0, \ldots, g_k$ by

$$g_i = \begin{cases} i = 0 & f(0, \ldots, 0) \\ i < k & f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots, x_k) f(x_1, \ldots, x_k)^{-1} \\ i = k & f(x_1, \ldots, x_k) f(x_0 +_F x_1, x_2, \ldots)^{-1} f(x_0, x_1 +_F x_2, \ldots) f(x_0, x_1, x_3, \ldots)^{-1} \end{cases}$$

Let I be the ideal in R generated by all the coefficients of all the power series $g_i - 1$. It is not hard to check $\operatorname{Spec}(R/I)$ has the universal property that defines $\underline{C}^k(G, \mathbb{G}_m)$.

n-cocycles in algebraic geometry

From definition we can make n-cocycles a sheaf as the following: let X, Y are commutative momoid fppf sheaves over S, we define $\underline{C}^k(X, Y)(T) = C^k(X_T, Y_T)$. It is actually a representable commutative monoid sheaf in $Sh(Sch/S)_{fppf}$ in certain case.

Proposition

Let G be a formal group over a scheme S. Then for all k, the functor $\underline{C}^k(G, \mathbb{G}_m)$ is an S-affine commutative group scheme.

Proof: It suffices to work $k \geq 1$ and locally on S, so we may assume $S = \operatorname{Spec}(R)$ and choose a coordinate x on G. We define power series $g_0, \ldots, g_k$ by

$$g_i = \begin{cases} i = 0 & f(0, \ldots, 0) \\ i < k & f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots, x_k) f(x_1, \ldots, x_k)^{-1} \\ i = k & f(x_1, \ldots, x_k) f(x_0 +_F x_1, x_2, \ldots)^{-1} f(x_0, x_1 +_F x_2, \ldots) f(x_0, x_1, x_3, \ldots)^{-1} \end{cases}$$

Let I be the ideal in R generated by all the coefficients of all the power series $g_i - 1$. It is not hard to check $\operatorname{Spec}(R/I)$ has the universal property that defines $\underline{C}^k(G, \mathbb{G}_m)$.

Even spaces

Definition

We say a space X to be "even" iff $H_*(X)$ is concentrated in even degrees and $H_n(X)$ is free abelian for all n.

Lemma (Hatcher 4C.1)

If X is even and simply-connected, then there exists a CW approximation $W \to X$ such that W only consists of cells of even degrees.

Proposition

(1) Let E be an EWP commutative ring (phantom-)spectrum. Then for any even space X, the A-T spectral sequence $H_*(X; E_*) \implies E_*(X)$ collapses . Therefore $E_*(X)$ is a free E_*-module and $E^*(X) \to Hom^*_{E_*}(E_*X, E_*)$ is bijective.

(2) The $E_0(X)$ is a cocommutative E_0-coalgebra by kunneth theorem. If X is an even H-space, we define $X_E = \operatorname{Spf} E^0 X$, then the natural Cartier morphism $\operatorname{Spec} E_0 X \to \underline{Hom}_{Grp/E}(X_E, \mathbb{G}_{m,E})$ is isomorphic, which is the special Cartier duality.

Definition

We say a space X to be "even" iff $H_*(X)$ is concentrated in even degrees and $H_n(X)$ is free abelian for all n.

Lemma (Hatcher 4C.1)

If X is even and simply-connected, then there exists a CW approximation $W \to X$ such that W only consists of cells of even degrees.

Proposition

(1) Let E be an EWP commutative ring (phantom-)spectrum. Then for any even space X, the A-T spectral sequence $H_*(X; E_*) \implies E_*(X)$ collapses . Therefore $E_*(X)$ is a free E_*-module and $E^*(X) \to Hom^*_{E_*}(E_*X, E_*)$ is bijective.
(2) The $E_0(X)$ is a cocommutative E_0-coalgebra by kunneth theorem. If X is an even H-space, we define $X_E = \operatorname{Spf} E^0X$, then the natural Cartier morphism $\operatorname{Spec} E_0X \to \underline{Hom}_{Grp/E}(X_E, \mathbb{G}_{m,E})$ is isomorphic, which is the special Cartier duality.

Proposition

Let X be an even commutative H-space, we have the following diagram in $CMon(Sets)$ for any $k \geq 0$,

$$\begin{array}{ccccc}
C^k(P,X) & \longrightarrow & C^k_{E_0-CcoAl}(E_0P, E_0X) & \dashrightarrow & Hom_{Mon/E}(X^E, \underline{C}^k(P_E, \mathbb{G}_{m,E})) \\
 & & \downarrow & \searrow & \downarrow \\
 & & \underline{C}^k(P_E, \mathbb{M}_{m,E})(\mathrm{Spec}\, E_0X) & \hookleftarrow & \underline{C}^k(P_E, \mathbb{G}_{m,E})(\mathrm{Spec}\, E_0X)
\end{array}$$

where $P = \mathbb{C}P^\infty$ is with the H-structure by tensor product of line bundles, and where $P_E = \mathrm{Spf}\, E^0P$, $X^E = \mathrm{Spec}\, E_0X$. The dashed liftings exist only when $k \geq 1$ or X is an H-group, and in those 2 cases all cocycle sets above are abelian groups.

Apply it to $\rho_t \in C^t(P, BU\langle 2t\rangle)$, we get morphisms of commutative group schemes over $\mathrm{Spec}(E_0)$ for all $t \geq 0$

$$f_t : \mathrm{Spec}\, E_0BU\langle 2t\rangle \to \underline{C}^k(P_E, \mathbb{G}_{m,E}).$$

The classical theory about the complex orientation tells us f_0 and f_1 are isomorphisms. Furthermore, we have the following

Theorem (Ando-Hopkins-Strickland)

The morphism $f_k : \operatorname{Spec} E_0 BU\langle 2k\rangle \to \underline{C}^k(P_E, \mathbb{G}_{m,E})$ is an isomorphism of group schemes over $\operatorname{Spec} E_0$ when $0 \leq k \leq 3$.

This theorem is, actually, the most technical part in [AHS], which involves amounts of calculus in both algebraic topology and algebraic geometry.
Now let us delay the proof and consider the following definition

Definition

If G and T are abelian group objects, and if $k \geq 0$ and $f \in C^k(G, T)$, then let $\delta(f) \in C^{k+1}(G, T)$ be the map given by the formula for $k \geq 1$

$$\delta(f)(a_0, \ldots, a_k) = f(a_0, a_2, \ldots, a_k) + f(a_1, a_2, \ldots, a_k) - f(a_0 + a_1, a_2, \ldots, a_k).$$

For $k = 0$, the map should be $\delta(f)(a) = f(0) - f(a)$

The classical theory about the complex orientation tells us f_0 and f_1 are isomorphisms. Furthermore, we have the following

Theorem (Ando-Hopkins-Strickland)

The morphism $f_k : \operatorname{Spec} E_0 BU\langle 2k\rangle \to \underline{C}^k(P_E, \mathbb{G}_{m,E})$ *is an isomorphism of group schemes over* $\operatorname{Spec} E_0$ *when* $0 \leq k \leq 3$.

This theorem is, actually, the most technical part in [AHS], which involves amounts of calculus in both algebraic topology and algebraic geometry.
Now let us delay the proof and consider the following definition

Definition

If G and T are abelian group objects, and if $k \geq 0$ and $f \in C^k(G,T)$, then let $\delta(f) \in C^{k+1}(G,T)$ be the map given by the formula for $k \geq 1$

$$\delta(f)\left(a_0, \ldots, a_k\right) = f\left(a_0, a_2, \ldots, a_k\right) + f\left(a_1, a_2, \ldots, a_k\right) - f\left(a_0 + a_1, a_2, \ldots, a_k\right).$$

For $k = 0$, the map should be $\delta(f)(a) = f(0) - f(a)$

δ-maps

It is clear that δ generalizes to abelian groups in any category with products. We leave it to the reader to verify the following.

Proposition

For $k \geq 0$, the map δ induces a homomorphism of groups

$$\delta : C^k(G, T) \to C^{k+1}(G, T)$$

Moreover, if G and T are formal groups over a scheme S, then δ induces a homomorphism of group schemes $\delta : \underline{C}^k(G, T) \to \underline{C}^{k+1}(G, T)$.

Proposition

The map ρ_t is contained in the subgroup $C^t(P, BU\langle 2t\rangle)$ of bu $bu^{2t}\left(P^t\right)$ and satisfies

$$v_* \rho_{t+1} = \delta\left(\rho_t\right) \in C^{t+1}(P, BU\langle 2t\rangle).$$

δ-maps

Proof of the proposition above: When $t \geq 1$, $v_* \rho_{t+1} = v \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_{t+1}$ and $\delta(\rho_t) = x_0 \cdot x_2 \cdot \ldots \cdot x_{t+1} + x_1 \cdot x_2 \cdot \ldots \cdot x_{t+1} - (x_0 +_{bu} x_1) \cdot x_2 \cdot \ldots \cdot x_{t+1} = v \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_{t+1}$.

$$= [x_0 + x_1 - (x_0 + x_1 - v \cdot x_1 x_2)] \cdot x_2 \cdot \ldots \cdot x_{t+1} = v \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_{t+1}$$

, i.e. $v_* \rho_{t+1} = \delta(\rho_t) \in C^{t+1}(P, BU\langle 2t \rangle)$.

Corollary

By the fact $v_* \rho_{t+1} = \delta(\rho_t) \in C^{t+1}(P, BU\langle 2t \rangle)$ *, we have the following diagram*

$$\begin{array}{ccc} BU\langle 2t \rangle^E & \xrightarrow{f_t} & \underline{C}^t(P_E, \mathbb{G}_{m,E}) \\ \downarrow & t \geq 0 & \downarrow \delta \\ BU\langle 2t+2 \rangle^E & \xrightarrow{f_{t+1}} & \underline{C}^{t+1}(P_E, \mathbb{G}_{m,E}) \end{array}$$

Definition

Suppose that $k \geq 1$ and G is an abelian big-Zariski-sheaf over S. Given a subset $I \subseteq \{1, \ldots, k\}$, we define $\sigma_I : G_S^k \to G$ by $\sigma_I(a_1, \ldots, a_k) = \sum_{i \in I} a_i$, and we write $\mathcal{L}_I = \sigma_I^* \mathcal{L}$, which is a line bundle over G_S^k. We also define the line bundle $\Theta^k(\mathcal{L})$ over G_S^k by the formula

$$\Theta^k(\mathcal{L}) \overset{\text{def}}{=} \bigotimes_{I \subset \{1,\ldots,k\}} (\mathcal{L}_I)^{(-1)^{|I|}}$$

Finally, we define $\Theta^0(\mathcal{L}) = \mathcal{L}$ For example we have

$$\Theta^0(\mathcal{L})_a = \mathcal{L}_a \,,\ \Theta^1(\mathcal{L})_a = \frac{\mathcal{L}_0}{\mathcal{L}_a} \,,\ \Theta^2(\mathcal{L})_{a,b} = \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b}}{\mathcal{L}_a \otimes \mathcal{L}_b}$$

$$\Theta^3(\mathcal{L})_{a,b,c} = \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c \otimes \mathcal{L}_{a+b+c}}$$

We observe three facts about these bundles.
(i) $\Theta^k(\mathcal{L})$ has a natural rigid structure for $k > 0$.
(ii) For each permutation $\sigma \in \Sigma_k$, there is a canonical isomorphism

$$\xi_\sigma : \pi_\sigma^* \Theta^k(\mathcal{L}) \cong \Theta^k(\mathcal{L})$$

where $\pi_\sigma : G_S^k \to G_S^k$ permutes the factors. Moreover, these isomorphisms compose in the obvious way.
(iii) There is a canonical identification (of rigid line bundles over G_S^{k+1})
$\Theta^k(\mathcal{L})_{a_1,a_2,\ldots} \otimes \Theta^k(\mathcal{L})^{-1}_{a_0+a_1,a_2,\ldots} \otimes \Theta^k(\mathcal{L})_{a_0,a_1+a_2,\ldots} \otimes \Theta^k(\mathcal{L})^{-1}_{a_0,a_1,\ldots} \cong 1$

Definition

A Θ^k-structure on a line bundle $\mathcal{L}$ over a group G is a trivialization s of the line bundle $\Theta^k(\mathcal{L})$ such that
(i) for $k > 0$, s is a rigid section;
(ii) s is symmetric in the sense that for each $\sigma \in \Sigma_k$, we have $\xi_\sigma \pi_\sigma^* s = s$;
(iii) the section
$s\left(a_1, a_2, \ldots\right) \otimes s\left(a_0 + a_1, a_2, \ldots\right)^{-1} \otimes s\left(a_0, a_1 + a_2, \ldots\right) \otimes s\left(a_0, a_1, \ldots\right)^{-1}$
corresponds to 1 under the isomorphism above.

n-cocycles of a line bundle

A Θ^3-structure on a line bundles is called by a cubical structure.

Definition

We write $C^k(G;\mathcal{L})$ for the set of Θ^k-structures on $\mathcal{L}$ over G. Note that $C^0(G;\mathcal{L})$ is just the set of trivializations of $\mathcal{L}$, and $C^1(G;\mathcal{L})$ is the set of rigid trivializations of $\Theta^1(\mathcal{L})$. We also define a functor from rings to sets by

$$\underline{C}^k(G;\mathcal{L})(R) = \left\{(u,f) \mid u : \mathrm{spec}(R) \to S, f \in C^k_{\mathrm{spec}(R)}(u^*G; u^*\mathcal{L})\right\}$$

Remark

Note that for the trivial line bundle $\mathcal{O}_G$, the set $C^k(G;\mathcal{O}_G)$ reduces to that of the group $\mathbb{C}^k(G,\mathbb{G}_m)$ of cocycles introduced previously.

Torsors and Thom sheaves

For any two line bundles $\mathcal{L}_1, \mathcal{L}_2$, we have natural $C^k(G;\mathcal{L}_1) \times C^k(G;\mathcal{L}_2) \to C^k(G;\mathcal{L}_1 \otimes \mathcal{L}_2)$ by $(s_1, s_2) \mapsto s_1 \otimes s_2$. Consequently, let $\mathcal{L}_1$ be trivial, then we can get a natural group action $C^k(G;\mathbb{G}_m) \times C^k(G;\mathcal{L}) \to C^k(G;\mathcal{L})$ for any line bundle $\mathcal{L}$.

Proposition

If G is a formal group over S, and $\mathcal{L}$ is a line bundle over G trivializable Zariski locally on S, then the functor $\underline{C}^k(G;\mathcal{L})$ is a scheme, whose formation commutes with change of base. Moreover, $\underline{C}^k(G;\mathcal{L})$ is a torsor for $\underline{C}^k(G,\mathbb{G}_m)$.

Now turn to the topology.

Definition

Suppose that X is a finite even complex and V is a virtual complex vector bundle classified by a $X \to Z \times BU$. We write X^V for its Thom spectrum. The coaction of the Thom spectrum makes $E^0 X^V$ an $E^0 X$-module. By Thom isomorphism Zariski locally, it is a line bundle further.

Torsors and Thom sheaves

For any two line bundles $\mathcal{L}_1, \mathcal{L}_2$, we have natural $C^k(G;\mathcal{L}_1) \times C^k(G;\mathcal{L}_2) \to C^k(G;\mathcal{L}_1 \otimes \mathcal{L}_2)$ by $(s_1, s_2) \mapsto s_1 \otimes s_2$. Consequently, let $\mathcal{L}_1$ be trivial, then we can get a natural group action $C^k(G;\mathbb{G}_m) \times C^k(G;\mathcal{L}) \to C^k(G;\mathcal{L})$ for any line bundle $\mathcal{L}$.

Proposition

If G is a formal group over S, and $\mathcal{L}$ is a line bundle over G trivializable Zariski locally on S, then the functor $\underline{C}^k(G;\mathcal{L})$ is a scheme, whose formation commutes with change of base. Moreover, $\underline{C}^k(G;\mathcal{L})$ is a torsor for $\underline{C}^k(G,\mathbb{G}_m)$.

Now turn to the topology.

Definition

Suppose that X is a finite even complex and V is a virtual complex vector bundle classified by a $X \to Z \times BU$. We write X^V for its Thom spectrum. The coaction of the Thom spectrum makes E^0X^V an E^0X-module. By Thom isomorphism Zariski locally, it is a line bundle further.

Torsors and Thom sheaves

Proposition

Suppose that X is a finite complex and V is a virtual bundle over X. We shall write $\mathbb{L}(V)$ for line bundle $\widetilde{E^0 X^V}$, and $\mathbb{L}$ defines a functor from vector bundles over X to line bundles over X_E.
(i) If V and W are two virtual complex vector bundles over X then there is a natural isomorphism

$$\mathbb{L}(V \oplus W) \cong \mathbb{L}(V) \otimes \mathbb{L}(W)$$

and so $\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}$.
(ii) Moreover, if $f : Y \to X$ is a map of spaces, then there is a natural isomorphism $f^* \mathbb{L}(V) \cong \mathbb{L}(f^* V)$ of line bundles over Y_E.

If X is an (infinite) even complex and V is a virtual bundle classified by $f : X \to BU\langle 0 \rangle$, then $\mathbb{L}(V)$ is a quasi-coherent sheaf on $\mathrm{Spf}\, E^0 X$ by taking colimits. Moreover, the proposition above also applies for infinite even complex X.

Proposition

Suppose that X is a finite complex and V is a virtual bundle over X. We shall write $\mathbb{L}(V)$ for line bundle $\widetilde{E^0 X^V}$, and $\mathbb{L}$ defines a functor from vector bundles over X to line bundles over X_E.
(i) If V and W are two virtual complex vector bundles over X then there is a natural isomorphism

$$\mathbb{L}(V \oplus W) \cong \mathbb{L}(V) \otimes \mathbb{L}(W)$$

and so $\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}$.
(ii) Moreover, if $f : Y \to X$ is a map of spaces, then there is a natural isomorphism $f^*\mathbb{L}(V) \cong \mathbb{L}(f^* V)$ of line bundles over Y_E.

If X is an (infinite) even complex and V is a virtual bundle classified by $f : X \to BU\langle 0 \rangle$, then $\mathbb{L}(V)$ is a quasi-coherent sheaf on $\operatorname{Spf} E^0 X$ by taking colimits. Moreover, the proposition above also applies for infinite even complex X.

Relation to MU<2k>

Lemma

Let $T(\rho_0) = \Sigma^\infty Th(\mathcal{L})$ is the Thom spectrum associated with $\rho_0 : P \to Z \times BU$ by the tautological bundle $\mathcal{L}$. Then the Thom sheaf $E^0 T(\rho_0)$ is naturally isomorphic to $\mathcal{I}(0) = ker(E^0 P \to E^0)$ in $Qcoh(P_E)$. This isomorphism is induced by a homotopy equivalence of P_+-comodule pointed spaces $P \to Th(\mathcal{L})$.

For $1 \le i \le k$, let L_i be the line bundle over the i factor of P^k. Recall that the map $\rho_k : P^k \to BU\langle 2k\rangle$ pulls the tautological virtual bundle over $BU\langle 2k\rangle$ back to the bundle

$$V = \bigotimes_i (1 - L_i)$$

Passing to Thom spectra gives a map

$$(P^k)^V \to MU\langle 2k\rangle$$

which determines an element s_k of $E_0 MU\langle 2k\rangle \widehat{\otimes} E^0\left((P^k)^V\right)$.

Lemma

Let $T(\rho_0) = \Sigma^\infty Th(\mathcal{L})$ *is the Thom spectrum associated with* $\rho_0 : P \to Z \times BU$ *by the tautological bundle* $\mathcal{L}$*. Then the Thom sheaf* $E^0 T(\rho_0)$ *is naturally isomorphic to* $\mathcal{I}(0) = ker(E^0 P \to E^0)$ *in* $Qcoh(P_E)$*. This isomorphism is induced by a homotopy equivalence of* P_+*-comodule pointed spaces* $P \to Th(\mathcal{L})$*.*

For $1 \leq i \leq k$, let L_i be the line bundle over the i factor of P^k. Recall that the map $\rho_k : P^k \to BU\langle 2k\rangle$ pulls the tautological virtual bundle over $BU\langle 2k\rangle$ back to the bundle

$$V = \bigotimes_i (1 - L_i)$$

Passing to Thom spectra gives a map

$$(P^k)^V \to MU\langle 2k\rangle$$

which determines an element s_k of $E_0 MU\langle 2k\rangle \widehat{\otimes} E^0\left((P^k)^V\right)$.

Together with properties of $\mathbb{L}$ give an isomorphism

$$\mathbb{L}(V) \cong \Theta^k(\mathcal{I}(0))$$

of line bundles over P_E^k. With this identification, s_k is a section of the pull-back of $\Theta^k(\mathcal{I}(0))$ along the projection $MU\langle 2k\rangle^E \to S_E$.

Proposition

The section s_k is a Θ^k-structure, and hence an element of $\underline{C}^k(P_E; \mathcal{I}(0))(MU\langle 2k\rangle^E)$.
Proof: This is analogous to the case of ρ_k.

Let

$$MU\langle 2k\rangle^E \xrightarrow{g_k} \underline{C}^k(P_E; \mathcal{I}(0))$$

be the map classifying the Θ^k-structure s_k. We note that the isomorphism $BU\langle 2k\rangle^E \cong \underline{C}^k(P_E, \mathbb{G}_m)$ gives $\underline{C}^k(P_E; \mathcal{I}(0))$ the structure of a torsor for the group scheme $BU\langle 2k\rangle^E$ when $k \leq 3$. It is worth noting that an equivariant morphism between torsors automatically become an isomorphism. Actually, the g_k is the case.

Thom sheaves and Θ^k-structures

Together with properties of $\mathbb{L}$ give an isomorphism

$$\mathbb{L}(V) \cong \Theta^k(\mathcal{I}(0))$$

of line bundles over P_E^k. With this identification, s_k is a section of the pull-back of $\Theta^k(\mathcal{I}(0))$ along the projection $MU\langle 2k\rangle^E \to S_E$.

Proposition

The section s_k is a Θ^k-structure, and hence an element of $\underline{C}^k(P_E;\mathcal{I}(0))(MU\langle 2k\rangle^E)$.
Proof: This is analogous to the case of ρ_k.

Let

$$MU\langle 2k\rangle^E \xrightarrow{g_k} \underline{C}^k(P_E;\mathcal{I}(0))$$

be the map classifying the Θ^k-structure s_k. We note that the isomorphism $BU\langle 2k\rangle^E \cong \underline{C}^k(P_E,\mathbb{G}_m)$ gives $\underline{C}^k(P_E;\mathcal{I}(0))$ the structure of a torsor for the group scheme $BU\langle 2k\rangle^E$ when $k \leq 3$. It is worth noting that an equivariant morphism between torsors automatically become an isomorphism. Actually, the g_k is the case.

Proposition

The following diagram is commutative

$$\begin{array}{ccc} BU\langle 2k\rangle^E \times MU\langle 2k\rangle^E & \longrightarrow & \underline{C}^k(P_E;\mathbb{G}_{m,E}) \times \underline{C}^k(P_E;\mathcal{I}(0)) \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle^E & \longrightarrow & \underline{C}^k(P_E;\mathcal{I}(0)) \end{array}$$

which is concluded by the following naturality of coactions on Thom spectra

$$\begin{array}{ccc} (P^k)^V & \longrightarrow & P^k_+ \wedge (P^k)^V \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle & \longrightarrow & BU\langle 2k\rangle_+ \wedge MU\langle 2k\rangle \end{array}$$

Theorem (Ando-Hopkins-Strickland)

The morphism $MU\langle 2k\rangle^E \xrightarrow{g_k} \underline{C}^k(P_E;\mathcal{I}(0))$ *is an isomorphism of* $BU\langle 2k\rangle^E$*-torsors when* $0 \leq k \leq 3$.

Since $MU\langle 2k\rangle$ is a bounded-below even spectrum when $k \leq 3$, we have natural isomorphisms

$$[MU\langle 2k\rangle, E] = E^0(MU\langle 2k\rangle) \to Hom_{E_*}(E_*MU\langle 2k\rangle, E_*) = Hom_{E_0}(E_0MU\langle 2k\rangle, E_0)$$

and

$$[MU\langle 2k\rangle, E]_{ring} = Hom_{E_0-Al}(E_0MU\langle 2k\rangle, E_0) = MU\langle 2k\rangle^E(S^E).$$

Corollary (Orientations correspond Θ^k-structures)

When $k \leq 3$, *the isomorphism* g_k *induces a bijection*

$$[MU\langle 2k\rangle, E]_{ring} \to C^k(P_E;\mathcal{I}(0))(S^E).$$

Orientations and Θ^k-structures

Theorem (Ando-Hopkins-Strickland)

The morphism $MU\langle 2k\rangle^E \xrightarrow{g_k} \underline{C}^k(P_E;\mathcal{I}(0))$ is an isomorphism of $BU\langle 2k\rangle^E$-torsors when $0 \le k \le 3$.

Since $MU\langle 2k\rangle$ is a bounded-below even spectrum when $k \le 3$, we have natural isomorphisms
$[MU\langle 2k\rangle, E] = E^0(MU\langle 2k\rangle) \to Hom_{E_*}(E_*MU\langle 2k\rangle, E_*) = Hom_{E_0}(E_0MU\langle 2k\rangle, E_0)$
and

$$[MU\langle 2k\rangle, E]_{ring} = Hom_{E_0-Al}(E_0MU\langle 2k\rangle, E_0) = MU\langle 2k\rangle^E(S^E).$$

Corollary (Orientations correspond Θ^k-structures)

When $k \le 3$, the isomorphism g_k induces a bijection

$$[MU\langle 2k\rangle, E]_{ring} \to C^k(P_E;\mathcal{I}(0))(S^E).$$

Theorem (Theorem of the cube)

Let $X \to S$ be an abelian scheme over S. Then for any $\mathcal{L} \in Pic(X)$, the $\Theta^3(\mathcal{L}) \cong p^*\mathcal{M}$ for some $\mathcal{M} \in Pic(S)$ where p denote the projection $X_S \times X_S \times_S X \to S$.
Furthermore, $\mathcal{O}_S \cong e^*\Theta^3(\mathcal{L})$ is naturally rigidificated, so $\mathcal{M} \cong e^*p^*\mathcal{M} \cong e^*\Theta^3(\mathcal{L}) \cong \mathcal{O}_S$ is trivial, and hence $\Theta^3(\mathcal{L})$ is also trivial.

Lemma

Let $p : X \to S$ be a proper smooth morphism with geometrically connected fibers, then
(i) The natural $\mathcal{O}_S \to p_*\mathcal{O}_X$ is isomorphic;
(ii) Let $e : S \to X$ be a section, and let $\mathcal{L}_1, \mathcal{L}_2$ be trivializable line bundles on X, then

$$Hom_{\mathcal{O}_X}(\mathcal{L}_1, \mathcal{L}_2) \to Hom_{\mathcal{O}_S}(e^*\mathcal{L}_1, e^*\mathcal{L}_2)$$

is bijective.

Theorem (Unique cubical structure for abelian schemes)

Let $p : X \to S$ be an abelian scheme over S. Then for any $\mathcal{L} \in Pic(X)$, there exists exactly one Θ^3-structure on $\mathcal{L}$.

Proof: Since $Hom_{\mathcal{O}_{X^3}}(\mathcal{O}_{X^3}, \Theta^3(\mathcal{L})) \to Hom_{\mathcal{O}_S}(\mathcal{O}_S, e^*\Theta^3(\mathcal{L}))$ is bijective by lemma above. The natural rigidification $\mathcal{O}_S \xrightarrow{1} e^*\Theta^3(\mathcal{L})$ determines unique trivialization $u : \mathcal{O}_{X^3} \to \Theta^3(\mathcal{L})$. Recall the axioms of cubical structures:
(i) $s(0) = 1$;
(ii) $s(a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}) = s(a_1, a_2, a_3)$ is symmetric for any $\sigma \in \Sigma_3$;
(iii) the section
$s\left(a_1, a_2, a_3\right) \otimes s\left(a_0 + a_1, a_2, a_3\right)^{-1} \otimes s\left(a_0, a_1 + a_2, a_3\right) \otimes s\left(a_0, a_1, a_3\right)^{-1} = 1.$

However, all conditions automatically hold for u by $u(0) = 1$ when we pullback to S along e, which means u is exactly the unique cubical structure.

Theorem (Unique cubical structure for abelian schemes)

Let $p : X \to S$ be an abelian scheme over S. Then for any $\mathcal{L} \in Pic(X)$, there exists exactly one Θ^3-structure on $\mathcal{L}$.

Proof: Since $Hom_{\mathcal{O}_{X^3}}(\mathcal{O}_{X^3}, \Theta^3(\mathcal{L})) \to Hom_{\mathcal{O}_S}(\mathcal{O}_S, e^*\Theta^3(\mathcal{L}))$ is bijective by lemma above. The natural rigidification $\mathcal{O}_S \xrightarrow{1} e^*\Theta^3(\mathcal{L})$ determines unique trivialization $u : \mathcal{O}_{X^3} \to \Theta^3(\mathcal{L})$. Recall the axioms of cubical structures:
(i) $s(0) = 1$;
(ii) $s(a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}) = s(a_1, a_2, a_3)$ is symmetric for any $\sigma \in \Sigma_3$;
(iii) the section
$s\left(a_1, a_2, a_3\right) \otimes s\left(a_0 + a_1, a_2, a_3\right)^{-1} \otimes s\left(a_0, a_1 + a_2, a_3\right) \otimes s\left(a_0, a_1, a_3\right)^{-1} = 1$.

However, all conditions automatically hold for u by $u(0) = 1$ when we pullback to S along e, which means u is exactly the unique cubical structure.

Proposition

Let $E \to F$ be a ring (phantom-)morphism between EWP ring (phantom-)spectra, and $MU\langle 2k\rangle \to E$ and $MU\langle 2k\rangle \to F$ be two orientations. Then

$$\begin{array}{ccc} & MU\langle 2k\rangle & \\ \swarrow & & \searrow \\ E & \longrightarrow & F \end{array}$$

commutes if and only if

$$\begin{array}{ccc} S^F & \longrightarrow & S^E \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle^F & \longrightarrow & MU\langle 2k\rangle^E \end{array}$$

commutes for the corresponding sections.

Theorem

(I) For any elliptic cohomology theories E we have natural σ-orientation $MU\langle 6\rangle \to E$.
(II) The σ-orientations commute for any morphism of elliptic cohomology theories $E \to F$ with morphism $C_1 \to C_2$ of elliptic curves.

$$\begin{array}{ccc} & MU\langle 2k\rangle & \\ \swarrow & & \searrow \\ E & \longrightarrow & F \end{array}$$

commutes by

$$\begin{array}{ccccccc} MU\langle 6\rangle^F & \xrightarrow{\simeq} & \underline{C}^3(P_F;\mathcal{I}(0)) & \longleftarrow & \underline{C}^3(C_1;\mathcal{I}(0)) & \xleftarrow{\simeq} & S^F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ MU\langle 6\rangle^E & \xrightarrow{\simeq} & \underline{C}^3(P_E;\mathcal{I}(0)) & \longleftarrow & \underline{C}^3(C_2;\mathcal{I}(0)) & \xleftarrow{\simeq} & S^E \end{array}$$

A sketch of proof of A-H-S theorem

For $k = 2$, $BU\langle 2k\rangle = BSU$. Consider the fiber sequence $BSU \to BU \xrightarrow{det} P$, which induces the following diagram of affine group schemes over S^E

$$\begin{array}{ccccc} P^E & \longrightarrow & BU^E & \longrightarrow & BSU^E \\ \downarrow & & \downarrow & & \downarrow \\ \underline{Hom}_{Grp/E}(P_E, \mathbb{G}_{m,E}) & \xrightarrow{f \mapsto 1/f} & \underline{C}^1(P_E, \mathbb{G}_{m,E}) & \xrightarrow{\delta} & \underline{C}^2(P_E, \mathbb{G}_{m,E}) \end{array}$$

The simplest, also important, example is $E = HP = \bigvee_{i\in\mathbb{Z}} \Sigma^{2i} HZ$.