

Formal Moduli Problems

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Applications





Formal Moduli Problems



Deformation Context

Definition

A deformation context is a pair $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$, where \mathcal{A} is a presentable ∞ -category with finite limits and E is a set of objects of $\text{Stab}(\mathcal{A})$.

1. A morphism in \mathcal{A} is elementary if it is a pull-back of $* \rightarrow \Omega^{\infty-n} E_\alpha$.
2. A morphism in \mathcal{A} is small if it can be written as a finite sequence of elementary morphisms.
3. An object A is artinian(small) if the morphism $A \rightarrow *$ is small.

Example

If $C = D(k)$, which is already stable, in this context, we can consider the spectrum object $E = (k[n+1])_{n \in \mathbb{Z}}$.

Formal Moduli Problems

Definition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. A formal moduli problem is a functor $X : \mathcal{A}^{\text{art}} \rightarrow \mathcal{S}$ satisfying the following pair of conditions:

1. The space $X(*)$ is contractible.
2. Let σ

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \phi \\ A & \longrightarrow & B \end{array}$$

be a pullback diagram in \mathcal{A}^{art} such that ϕ is small, then $X(\sigma)$ is pullback diagram in \mathcal{S} .

Example

Let $B \in \mathcal{A}^{\text{art}}$,

$$\text{Spf}(B) : \mathcal{A}^{\text{art}} \rightarrow \mathcal{S}, \quad A \mapsto \text{Map}_{\mathcal{A}}(B, A)$$

Tangent Complex

Definition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, $Y : \mathcal{A}^{\text{art}} \rightarrow \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, the tangent complex of Y at α is the following composite functor

$$\mathcal{S}_*^{\text{fin}} \xrightarrow{E_\alpha} \mathcal{A}^{\text{art}} \xrightarrow{Y} \mathcal{S}.$$

Proposition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be deformation context and let $u : X \rightarrow Y$ be a map of formal moduli problems. Suppose that u induces an equivalence of tangent complexes

$$X(E_\alpha) \rightarrow Y(E_\alpha)$$

for each $\alpha \in T$. Then u is an equivalence.

Weak Deformation Theory

Definition

A weak deformation theory for a deformation context $(\mathcal{A}, \{E_\alpha\})$ is a functor $\mathcal{D} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ satisfying the following conditions

1. The ∞ -category is presentable.
2. The functor admits a left adjoint $\mathcal{D}' : \mathcal{B} \rightarrow \mathcal{A}^{op}$.
3. There exists a full subcategory $\mathcal{B}_0 \subset \mathcal{B}$ satisfying the following conditions:
 - ▣ For every $K \in \mathcal{B}_0$, the unit map $K \rightarrow \mathcal{D}\mathcal{D}'K$ is an equivalence.
 - ▣ \mathcal{B}_0 contains the initial object $\emptyset \in \mathcal{B}$.
 - ▣ For every $\alpha \in T$ and every $n \geq 1$, there exists an object $K_{\alpha,n} \in \mathcal{B}_0$ and an equivalence $\Omega^{\infty-n}E_\alpha \simeq \mathcal{D}'K_{\alpha,n}$.
 - ▣ For every pushout diagram

$$\begin{array}{ccc} K_\alpha & \longrightarrow & k \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & K' \end{array}$$

If K belongs to \mathcal{B}_0 , then K' also belongs to \mathcal{B}_0 .

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, $\mathcal{D} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ a weak deformation theory, and $j : \mathcal{B} \rightarrow \text{Fun}(\mathcal{B}^{op}, \mathcal{S})$ be the Yoneda embedding. Then

1. For every $B \in \mathcal{B}$, the composition

$$\mathcal{A}^{\text{art}} \subset \mathcal{A} \xrightarrow{\mathcal{D}} \mathcal{B}^{op} \xrightarrow{j(B)} \mathcal{S}$$

is a formal moduli problem.

2. The construction $B \mapsto (j(B) \circ \mathcal{D})|_{\mathcal{A}^{\text{art}}}$ determine a functor

$$\Psi : \mathcal{B} \rightarrow \text{Moduli}^{\mathcal{A}}$$

3. The diagram

$$\begin{array}{ccc} \mathcal{A}^{op} & \xrightarrow{\text{Spf}} & \text{Moduli}^{\mathcal{A}} \\ & \searrow \mathcal{D} & \nearrow \Psi \\ & \mathcal{B} & \end{array}$$

commutes up to homotopy.



Weak Deformation Theory

Proposition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathcal{D} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ a weak deformation theory. $\mathcal{B}_0 \subset \mathcal{B}$ be a full subcategory satisfying the condition above, then

1. \mathcal{D} carries final objects of \mathcal{A} to initial objects of \mathcal{B} .
2. If $A = \mathcal{D}'(K)$ for some $K \in \mathcal{B}_0$. Then the unit map $A \rightarrow \mathcal{D}'\mathcal{D}(A)$ is an equivalence in \mathcal{A} .
3. If $A \in \mathcal{A}^{\text{art}}$, $\mathcal{D}(A) \in \mathcal{B}_0$ and the unit map $A \rightarrow \mathcal{D}'\mathcal{D}(A)$ is an equivalence in \mathcal{A} .
4. If we have a pullback diagram σ

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \phi \\ A & \longrightarrow & B \end{array}$$

in \mathcal{A} where $A, B \in \mathcal{A}^{\text{art}}$ and the morphism ϕ is small. Then $\mathcal{D}(\sigma)$ is a pushout diagram in \mathcal{B} .

Deformation Theory

Lemma

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathcal{D} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ a weak deformation theory. For each $\alpha \in T$ and each $K \in \mathcal{B}$, the composite map

$$\mathcal{S}_*^{fin} \xrightarrow{E_\alpha} \mathcal{A} \xrightarrow{\mathcal{D}} \mathcal{B}^{op} \xrightarrow{j(K)} \mathcal{S}$$

is reduced and excisive and therefore can be identified with a spectrum which we will denote by $e_\alpha(K)$. This determines a functor $e_\alpha : \mathcal{B} \rightarrow \text{Sp}$.

Definition

A deformation theory for $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ is a weak deformation theory $\mathcal{D} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ satisfying the following condition: For each $\alpha \in T$, the functor $e_\alpha : \mathcal{B} \rightarrow \text{Sp}$ preserves small sifted colimits. Moreover, a morphism f in \mathcal{B} is an equivalence if and only if each $e_\alpha(f)$ is an equivalence of spectra.

Formal Moduli Problems

Main Theorem

Given a deformation context $(\mathcal{A}^{op}, \{E_\alpha\}_{\alpha \in T})$ and a deformation theory (Koszul duality context)

$$\mathcal{D} : \mathcal{A}^{op} \rightleftarrows \mathcal{B} : \mathcal{D}' ,$$

Then the functor

$$\Psi : \mathcal{B} \rightarrow \text{Moduli}^{\mathcal{A}}$$

is an equivalence of ∞ -category.



Sketch of Proof

Lemma

Let $(A, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathcal{D} : \mathcal{A}^{op} \rightarrow \mathcal{B}$ be a deformation theory. For every Artinian object $A \in \mathcal{A}^{art}$, $\mathcal{D}(A)$ is a compact object of the ∞ -category \mathcal{B} .

The functor $\Psi : \mathcal{B} \rightarrow \text{Moduli}^A \subset \text{Fun}(\mathcal{A}^{art}, \mathcal{S})$ is defined by

$$\Psi(K)(A) = \text{Map}_{\mathcal{B}}(\mathcal{D}(A), K)$$

Ψ preserves small limits. And Ψ preserves filtered colimits and is therefore accessible. So by the ∞ -categorical adjoint functor theorem, Ψ admits a left adjoint Φ . To prove that Ψ is an equivalence, it will suffice to show that

1. The functor Ψ is conservative.
2. The unit transformation $u : \text{Id} \rightarrow \Psi \circ \Phi$ is an equivalence.



Proof of Conservative

Let $f : K \rightarrow K'$ in \mathcal{B} , such that $\Psi(f)$ is an equivalence.

$$\mathrm{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_{\alpha}), K) \simeq \Psi(K)\mathcal{D}(\Omega^{\infty-n}E_{\alpha})$$

$$\mathrm{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_{\alpha}), K') \simeq \Psi(K')\mathcal{D}(\Omega^{\infty-n}E_{\alpha})$$

It follows that $e_{\alpha}(K) \simeq e_{\alpha}(K')$. Since the functors are jointly conservative, we conclude that f is an equivalence.



Proof of Equivalence

To prove that $X \rightarrow \Psi \circ \Phi(X)$ is an equivalence, by the proposition of tangent complex, it suffice to show that for each $\alpha \in T$, the induced map

$$\theta : X(E_\alpha) \rightarrow (\Psi \circ \Phi)(X)(E_\alpha) \simeq e_\alpha(\Phi X)$$

is equivalence of spectra.

Every formal moduli problems admits a smooth hypercovering by "affine" objects.

Proposition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $X : \mathcal{A}^{\text{art}} \rightarrow \mathcal{S}$ be a formal moduli problem. Then there exists a simplicial objects X_\bullet in $\text{Moduli}_{/X}^{\mathcal{A}}$ with the following properties:

1. Each X_n is prorepresentable.
2. For each $n \geq 0$, let $M_n(X_\bullet)$ denote the matching object of the simplicial object X_\bullet . Then the canonical map $X_n \rightarrow M_n(X_\bullet)$ is smooth.

In particular, X is equivalent to the geometric realization $|X_\bullet|$ in $\text{Fun}(\mathcal{A}^{\text{art}}, \mathcal{S})$.

$$\theta : X(E_\alpha) \rightarrow (\Psi \circ \Phi)(X)(E_\alpha) \simeq e_\alpha(\Phi X)$$

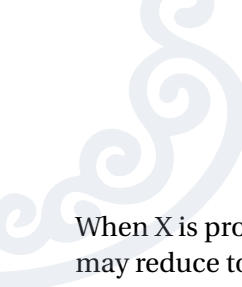
Choose a simplicial object X_\bullet of $\text{Moduli}_{/X}^A$ satisfying the above proposition. For each $a \in A^{\text{art}}$.

1. $X_\bullet(a)$ is an hypercovering of $X(A)$, $|X_\bullet(A)| \rightarrow X(A)$ is a homotopy equivalence.
2. X is a colimit of the diagram X_\bullet in the ∞ -category $\text{Fun}(A^{\text{art}}, \mathcal{S})$.
3. Similarly, $X(E_\alpha)$ is equivalent to the geometric realization $|X_\bullet(E_\alpha)|$.
4. Since Φ preserves small colimits and e_α preserves sifted colimits.

$$e_\alpha(\Phi(X)) \simeq e_\alpha(\Phi|X_\bullet|) \simeq |e_\alpha(\Phi X_\bullet)|.$$

5. It follows that θ is a geometric realization of a simplicial morphism $\theta_\bullet : X_\bullet(E_\alpha) \rightarrow e_\alpha(\Phi X_\bullet)$.
6. It suffices to prove that each θ_n is an equivalence.
7. Equivalent to prove that $X_n \rightarrow (\Psi \circ \Phi)(X_n)$ is an equivalence.





When X is prorepresentable, since Φ and Ψ both commutes with filtered colimits. We may reduce to the case $X = \mathrm{Spf}(A)$ for some $A \in \mathcal{A}^{\mathrm{art}}$. But $\Phi(\mathrm{Spf}(A)) = \mathcal{D}(A)$, it is equivalent to prove that for each $B \in \mathcal{A}^{\mathrm{art}}$, the map

$$\mathrm{Map}_{\mathcal{A}(A,B)} \rightarrow \mathrm{Map}_{\mathcal{B}}(\mathcal{D}(B), \mathcal{D}(A)) \simeq \mathrm{Map}_{\mathcal{A}}(A, \mathcal{D}'\mathcal{D}(B)).$$

This a consequence form above proposition.





Applications



Formal Moduli Problems in Different Graded Algebras

1. $\text{Cdga}_k^{\text{aug}}$ is the ∞ -category of augmented commutative differential graded algebras.
2. A morphism in $\text{Cdga}_k^{\text{aug}}$ is called elementary if it is a pullback of $k \rightarrow k \oplus k[n]$ for some $n \geq 1$, where $k \rightarrow k \oplus k[n]$ is the square zero extension of k by $k[n]$.
3. A morphism in $\text{Cdga}_k^{\text{aug}}$ is called small if it is a finite composition of elementary morphisms.
4. An object in $\text{Cdga}_k^{\text{aug}}$ is called small if the augmentation morphism $\epsilon : A \rightarrow k$ is small.

Proposition

An object $\text{Cdga}_k^{\text{aug}}$ is small if and only if the following conditions hold:

1. $H^n(A) = \{0\}$ for n positive and for n sufficiently negative.
2. All cohomology groups $H^n(A)$ are finite dimensional over k .
3. $H^0(A)$ is a local ring with maximal ideal m , and the morphism $H^0(A)/m \rightarrow k$ is an isomorphism.

Definition

A formal moduli problem is an ∞ -functor $X : (\text{Cdga})_k^{sm} \rightarrow \mathcal{S}$ satisfying the following two conditions:

1. $X(k)$ is contractible.
2. X preserves pull-back along small morphisms.

The second condition means that given a Cartesian diagram

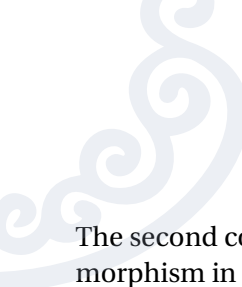
$$\begin{array}{ccc} N & \longrightarrow & A \\ \downarrow & & \downarrow \\ M & \longrightarrow & B \end{array}$$

in Cdga where $A \rightarrow B$ is small, then

$$\begin{array}{ccc} X(N) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(M) & \longrightarrow & X(B) \end{array}$$

is Cartesian.






The second condition is stable under composition and pullback. We can replace small morphism in the condition by $k \rightarrow k \oplus k[n]$.

Theorem

There is a equivalence of ∞ -categories $\mathrm{dgLie}_k \rightarrow \mathrm{Moduli}_k$.



Chevalley-Eilenberg Complex

For any differential graded Lie algebra \mathfrak{g} , we can construct the homological and cohomological Chevalley-Eilenberg complex CE_\bullet .

1. As vector space $CE_\bullet = S(\mathfrak{g}[1])$ is the graded symmetric algebra of $\mathfrak{g}[1]$. The differential is obtained by extending, as a degree graded coderivation. The complex CE_\bullet is actually counital, conilpotent cocommutative coalgebra object in the category of complexes.
2. CE^\bullet is the linear dual of $CE_\bullet(\mathfrak{g})$, it is an augmented cdga.

$$CE_\bullet(\mathfrak{g}) \simeq k \otimes_{U(\mathfrak{g})}^L k \simeq \mathrm{Tor}_\bullet^{U(\mathfrak{g})}(k, k)$$
$$CE^\bullet(\mathfrak{g}) \simeq \mathrm{RHom}_{U(\mathfrak{g})}(k, k) \simeq \mathrm{Ext}_{U(\mathfrak{g})}^\bullet(k, k)$$



The Chevalley-Eilenberg construction preserves weak equivalence, thus defining an functor

$$CE^\bullet : \text{Lie}_k^{op} \rightarrow \text{CAlg}_k^{aug}$$

This functor commutes with small colimits.

The ∞ -category Lie_k is presentable, so CE^\bullet admits a left adjoint. We denote this adjoint \mathcal{D} .

$$\mathcal{D} : \text{CAlg}_k^{aug} \rightleftarrows \text{Lie}_k^{op} : CE^\bullet$$

We define an ∞ -functor from Lie_k to $\text{Fun}(\text{CAlg}_k^{aug}, \mathcal{S})$

$$\Delta(\mathfrak{g}) = \text{Hom}_{\text{Lie}_k^{op}}(\mathfrak{g}, \mathcal{D}(-)) = \text{Hom}_{\text{Lie}_k}(\mathcal{D}(-), \mathfrak{g})$$



A differential graded Lie algebra L is good if there exists a finite chain $0 = L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_n = L$ such that each of these morphism appears in the pushout diagram

$$\begin{array}{ccc} \text{free } k[-n_i - 1] & \longrightarrow & L_i \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & L_{i+1} \end{array}$$

is

Lemma

If \mathfrak{g} is good, the counit morphism $\mathcal{DCE}^\bullet(\mathfrak{g})$ in Lie_k^{op} is an equivalence.

If we have a cartesian diagram


$$\begin{array}{ccc} N & \longrightarrow & k \\ \downarrow & & \downarrow \\ M & \longrightarrow & k \oplus k[n] \end{array}$$

where N and M are small, then

$$\begin{array}{ccc} \mathcal{D}(N) & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \mathcal{D}(M) & \longrightarrow & \mathcal{D}(k \oplus k[n]) \end{array}$$

is also cartesian in Lie_k^{op} and therefore




$$\begin{array}{ccc} \Delta(\mathfrak{g})(N) & \longrightarrow & \star \\ \downarrow & & \downarrow \\ \Delta(\mathfrak{g})(M) & \longrightarrow & \Delta(\mathfrak{g})(k \oplus k[n]) \end{array}$$

is also cartesian in \mathbf{sSet} . So Δ is an object of \mathbf{FMP}_l . Hence Δ factor through the category \mathbf{FMP}_k .



Formal Moduli Problem for Associative Algebras

Assume that k is a field, $X : \text{Alg}_k^{\text{art}} \rightarrow \mathcal{S}$ be a functor. We will say that X is a formal E_1 -moduli problem if it satisfies the following conditions:

1. $X(k)$ is contractible.
2. For every pullback diagram σ

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\text{Alg}_k^{\text{art}}$ where the underlying maps $\pi_0 R_0 \rightarrow \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Then $X(\sigma)$ is a pull back square.

Theorem

Let k be a field. Then there is an equivalence of ∞ -categories

$$\text{Alg}_k^{\text{aug}} \rightarrow \text{Moduli}_k^{(1)}.$$

Moduli Problem for E_n algebras

There is a diagram

$$\dots \rightarrow \text{Alg}_k^{(3)} \rightarrow \text{Alg}_k^{(2)} \rightarrow \text{Alg}_k^{(1)} \simeq \text{Alg}_k,$$

where $\text{Alg}_k^{(n)}$ denote the ∞ -category of E_n algebras over k .

We say that $A \in \text{Alg}_k^{(n)}$ is Artinian if its image in Alg_k is Artinian.

$X : \text{Alg}_k^{(n), \text{art}} \rightarrow \mathcal{S}$ be a functor. We will say that X is a formal E_n -moduli problem if it satisfies the following conditions:

1. $X(k)$ is contractible.
2. For every pullback diagram σ

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\text{Alg}_k^{(n), \text{art}}$ where the underlying maps $\pi_0 R_0 \rightarrow \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Then $X(\sigma)$ is a pull back square.



Theorem

Let k be a field. Then there is an equivalence of ∞ -categories

$$\mathrm{Alg}_k^{(n),aug} \rightarrow \mathrm{Moduli}_k^{(n)}.$$

Moreover, the diagram

$$\begin{array}{ccc} \mathrm{Alg}_k^{(n),aug} & \xrightarrow{\Psi} & \mathrm{Moduli}_k^n \\ \downarrow m_A & & \downarrow \Sigma^{-n}T \\ \mathrm{Mod}_k & \longrightarrow & \mathrm{Sp} \end{array}$$

commutes up to homotopy.

Deformation as Formal Moduli Problems

Given a smooth scheme Z over k , then formal deformation theory of Z deal with the equivalence classes of Cartesian diagrams

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(A) \end{array}$$

where A is a local artinian algebra with residue field k . This define a deformation functor Def_Z from the category of local artinian algebra to sets.

Theorem

When $A = k[t]/t^2$. There is a bijection between the isomorphism class of X over $\text{Spec}(k[t]/t^2)$ and the cohomology $H^1(Z, T_Z)$.

Let $f : X \rightarrow S$ be a scheme, and $t : S \rightarrow S'$ be a square zero infinitesimal thickening, which is morphism of scheme with the kernel

$$\mathcal{I} = \ker(\mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$$

satisfying $\mathcal{I}^2 = 0$. Given a \mathcal{O}_X -module \mathcal{G} , and a morphism $\mathcal{I} \rightarrow \mathcal{G}$ of \mathcal{O}_X module. We ask whether we can find a \mathcal{M} fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{M}(?) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array} \quad (1)$$

and what situation the solution is unique?



Theorem

In the situation above we have

1. There is a canonical element $\zeta \in \text{Ext}_{\mathcal{O}_X}^2(L_{X/S}, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to the above diagram.
2. If there exists a solution, then the set of isomorphism classes of solution is principal homogeneous under $\text{Ext}_{\mathcal{O}_X}^1(L_{X/S}, \mathcal{G})$.
3. Given a solution X' , the set of automorphisms of X' fitting into the diagram is canonically isomorphic to $\text{Ext}_{\mathcal{O}_X}^0(L_{X/S}, \mathcal{G})$



Deformation Theory in the Higher Categorical Case

Let k be field, \mathcal{C} be a stable k -linear ∞ -category, and $E \in \mathcal{C}$

$$\text{Def} : \text{Alg}^{\text{art}} \rightarrow \mathcal{S}$$

$$B \mapsto (\text{RMod}_B(\mathcal{C}) \times_{\mathcal{C}} E)^{\simeq}$$

Theorem

Let k be field, \mathcal{C} be a stable k -linear ∞ -category, and $E \in \mathcal{C}$. Let $\Psi : \text{Alg}_k^{\text{aug}} \rightarrow \text{Moduli}_k^{(1)}$ be the equivalence of ∞ -category of formal moduli problem. Then there is an equivalence of formal \mathbb{E}_1 -moduli problems

$$\text{Def}_E \simeq \Psi(k \oplus \text{End}(E)).$$