Goerss Hopkins obstruction theory

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1. The moduli space of realizations
2. Decomposition of $\mathcal{TM}_\infty(A)$
3. Obstruction theory

E_∞ Realization problem

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- A: an algebra in the category of E_*E-comodules.

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Q: Whether $\mathcal{TM}(A)$ is nonempty?

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Let C be an E_∞-operad. Then there exists an argumented simplicial operad $T \to C$ s.t.

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Let $sAlg_T$ be the category of simplicial T-algebras in spectra.

$$X \in sAlg_T \implies |X| \in Alg_{|T|}.$$

Remark

(1) E satisfies Adams condition: There is a simplicial model structure on $sAlg_T$ determined by a collection of spectra $\mathcal{P}$ generated by DE_α.

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Then

$$X \in \mathcal{TM}_\infty(A) \implies |X| \in \mathcal{TM}(A),$$

and the geometric realization functor gives a weak equivalence $\mathcal{TM}_\infty(A) \simeq \mathcal{TM}(A)$.

the spiral exact sequence

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The exact couple related to the above spectral sequence gives a long exact sequence called the spiral exact sequence:

$$\to \pi_{p-1,q+1}(E \wedge X) \to \pi_{p,q}(E \wedge X) \to \pi_p E_q X \to \pi_{p-2,q+1}(E \wedge X) \to$$

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If $X \in \mathcal{TM}_\infty(A)$, then $\pi_{p,*}(E \wedge X) \cong \Omega^p(A)$ for all $p \geq 0$.

Postnikov system for simplicial algebras in spectra

Definition

Let $X \in sAlg_T$. A Postnikov tower for X is a tower of simplicial T-algebras under X

$$X \to \cdots \to P_n X \to P_{n-1} X \to \cdots \to P_0 X,$$

s.t. for every $f : X \to P_n X$,

$$f_* : \pi_{i,*}(E \wedge X) \xrightarrow{\cong} \pi_{i,*}(E \wedge P_n X), \quad i \leq n,$$

and s.t. $\pi_{i,*}(E \wedge P_n X) = 0$ for $i > n$.

Decomposition of $\mathcal{TM}_\infty(A)$

Definition

Let $X \in sAlg_T$. We say that X is a potential n-stage for A if

$$\pi_i E_* X \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq n+1 \end{cases}$$

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$$\pi_{i,*}(E \wedge X) \cong \begin{cases} \Omega^i A & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

Let $\mathcal{TM}_n(A)$ be the moduli space of potential n-stages for A, then the Postnikov section induces

$$\mathcal{TM}_\infty(A) \to \cdots \to \mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A) \to \ldots \mathcal{TM}_0(A).$$

Now we can try to build $X \in \mathcal{TM}_\infty(A)$ inductively.

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(1) A simplicial T-algebra is of type B_A if $\pi_{0,*}(E \wedge X) \cong A$ and $\pi_{i,*}X = 0$ for $i > 0$, and for all simplicial T-algebras Y , the natural map

$$[Y, X] \to \hom_{E_* C}(\pi_0 E_* Y, A)$$

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(2) Given an A-module M, a morphism $X \to Y$ of T-algebras is of type $B_A(M, n)$ if X is type B_A, $\pi_{0,*}(E \wedge X) \to \pi_{0,*}(E \wedge Y)$ is an isomorphism, and

$$\pi_{i,*}(E \wedge Y) \cong \begin{cases} M & \text{if } i = n \\ 0 & \text{if } i \neq 0, n \end{cases}$$

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Proposition

There is a natural map $E_*B_A(M,n) \to K_A(M,n)$ which induces an isomorphism

$$\hom_{sAlg_T/B_A}(X, B_A(M,n)) \to \hom_{sAlg_{E_*T/E_*E}/A}(E_*(X), K_A(M,n)).$$

Here $K_A(M,n) = K(M,n) \rtimes A \in sAlg_{E_*T/E_*E}$.

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If $X \in \mathcal{TM}_n(A)$, $Y = P_{n-1}X$, then

$$\begin{array}{ccc} X & \longrightarrow & B_A \\ \downarrow & \ulcorner & \downarrow \\ Y & \xrightarrow{f} & B_A(\Omega^n A, n+1) \end{array}$$

Obstruction theory

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Proposition

If $Y \in \mathcal{TM}_{n-1}(A)$, then $X \in \mathcal{TM}_n(A)$ if and only if the map $E_*(Y) \to K_A(\Omega^n A, n+1)$ induced by f is a weak equivalence.

Let $Y \in \mathcal{TM}_{n-1}(A)$, then

$$\begin{array}{ccc} E_* Y & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ A \simeq P_0 E_* Y & \longrightarrow & K_A(\Omega^n A, n+2) \end{array}$$

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Corollary

There are obstructions $\theta_n \in \hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(M, n))$ to existance of a commutative S-algebra X with $E_*X \cong A$.

Let

$$\mathcal{H}^n(A; M) = \hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(M, n)),$$

$$\hat{\mathcal{H}}^n(A; M) = \mathcal{H}^n(A; M) \times_{Aut(A,M)} EAut(A, M).$$

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Theorem

The following is a homotopy pullback square:

$$\begin{array}{ccc} \mathcal{TM}_n(A) & \longrightarrow & BAut(A,\Omega^n A) \\ \downarrow & & \downarrow \\ \mathcal{TM}_{n-1}(A) & \longrightarrow & \hat{\mathcal{H}}^{n+2}(A;\Omega^n A) \end{array}$$