An overview of ∞-category and Higher Algebra

Jiacheng Liang

SUSTech Department of Mathematics

December 26, 2023

Motivation of the ∞-category

Motivation 1

The most significant motivation is to change the morphism **set** $Hom_{\mathcal{C}}(X, Y)$ in a category $\mathcal{C}$ to a topological **space** $Map_{\mathcal{C}}(X, Y)$. Then we can have higher morphisms $\pi_n Map_{\mathcal{C}}(X, Y)$.

For example when considering the category of spectra, we have $\pi_n Map_{\mathcal{C}}(X, Y) = [\Sigma^n X, Y]$.

Motivation 2

We want to **internalize** the category theory. In another way, we want to **characterize** a specific ∞-category by a universal property in the ∞-category of all ∞-categories $\mathcal{C}at_{\infty}$.

For instance, we will see the ∞-category of spaces $\mathcal{S}$ is "free generated" by the single-point space $* \in \mathcal{S}$.

Motivation of the ∞-category

Motivation 1

The most significant motivation is to change the morphism **set** $Hom_{\mathcal{C}}(X, Y)$ in a category $\mathcal{C}$ to a topological **space** $Map_{\mathcal{C}}(X, Y)$. Then we can have higher morphisms $\pi_n Map_{\mathcal{C}}(X, Y)$.

For example when considering the category of spectra, we have $\pi_n Map_{\mathcal{C}}(X, Y) = [\Sigma^n X, Y]$.

Motivation 2

We want to **internalize** the category theory. In another way, we want to **characterize** a specific ∞-category by a universal property in the ∞-category of all ∞-categories $\mathcal{C}at_{\infty}$.

For instance, we will see the ∞-category of spaces $\mathcal{S}$ is "free generated" by the single-point space $* \in \mathcal{S}$.

Extracting information from ∞-category

The most intuitive model for ∞-category is the $sSet$-enriched (or Top-enriched) category. Actually we have a Quillen equivalence $sSet_{Joyal} \rightleftarrows Cat_{\Delta}$.

Mapping spaces

There are at least 4 definitions of the mapping space $Map_{\mathcal{C}}(X, Y)$ in an ∞-category $\mathcal{C}$. But when we take their underlying $Ho(sSet_{Kan})$-enriched categories, all of them are the same, written as $\underline{h\mathcal{C}}$. ($* * *$ most important invariant)

How to extract useful and discard redundant information in certain circumstances is an art in ∞-category's world.

Example

For example when we want to show a functor between ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, it suffices to show $\underline{hF} : \underline{h\mathcal{C}} \to \underline{h\mathcal{D}}$ is equivalent.
But if we consider colimits in an ∞-category $\mathcal{C}$, we need more homotopy coherent information than those in $\underline{h\mathcal{C}}$. In this case, we can't reduce to $\underline{h\mathcal{C}}$.

Extracting information from ∞-category

The most intuitive model for ∞-category is the $sSet$-enriched (or Top-enriched) category. Actually we have a Quillen equivalence $sSet_{Joyal} \rightleftarrows Cat_{\Delta}$.

Mapping spaces

There are at least 4 definitions of the mapping space $Map_{\mathcal{C}}(X, Y)$ in an ∞-category $\mathcal{C}$. But when we take their underlying $Ho(sSet_{Kan})$-enriched categories, all of them are the same, written as $\underline{h\mathcal{C}}$. ($* * *$ most important invariant)

How to extract useful and discard redundant information in certain circumstances is an art in ∞-category's world.

Example

For example when we want to show a functor between ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, it suffices to show $\underline{hF} : \underline{h\mathcal{C}} \to \underline{h\mathcal{D}}$ is equivalent.
But if we consider colimits in an ∞-category $\mathcal{C}$, we need more homotopy coherent information than those in $\underline{h\mathcal{C}}$. In this case, we can't reduce to $\underline{h\mathcal{C}}$.

Why use the ∞-category?

Some phenomenons or propositions can't be stated clearly without ∞-category.

Example

(1) Chromatic convergence and chromatic pullback:

$$
\begin{array}{ccc}
 & & \vdots \\
 & \nearrow & \downarrow \\
 & & L_1 X \\
 & \nearrow & \downarrow \\
X & \longrightarrow & L_0 X
\end{array}
\qquad
\begin{array}{ccc}
L_n X & \longrightarrow & L_{K(n)} X \\
\downarrow & & \downarrow \\
L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X
\end{array}
$$

They are homotopy limits of **homotopy coherent diagrams** $N_*(\mathbb{Z}^{op}_{\geq 0}) \to Sp$ and $\Lambda^2_2 \to Sp$. However, classical framework only provides homotopy diagrams, which can't be used to take homotopy limit.

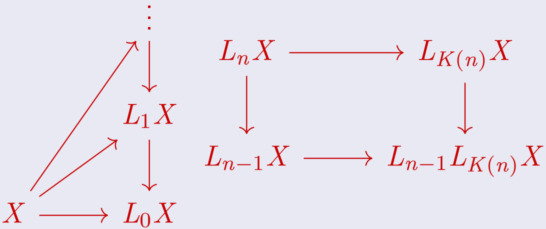
(2) Postnikov tower in the category of spaces $\mathcal{S}$ and its convergence are the same philosophy as above.

Why use the ∞-category?

Some phenomenons or propositions can't be stated clearly without ∞-category.

Example

(1) Chromatic convergence and chromatic pullback:



They are homotopy limits of **homotopy coherent diagrams** $N_*(\mathbb{Z}_{\geq 0}^{op}) \to Sp$ and $\Lambda_2^2 \to Sp$. However, classical framework only provides homotopy diagrams, which can't be used to take homotopy limit.

(2) Postnikov tower in the category of spaces $\mathcal{S}$ and its convergence are the same philosophy as above.

Why use the ∞-category?

Example

(2) Equivariant stable homotopy theory: there are plenty of model categories characterizing it, but all of their underlying ∞-category are equivalent with $Fun(BG, Sp)$, which is both simple and intuitive.
(3) In ∞-framework, the E_∞-operad is just commutative operad. And E_∞ spaces, E_∞-spectra are ∞-commutative monoid objects.
(4) We have all kinds of well-defined moduli spaces, like $Fun^{\otimes}(C, D)$, $Fun^{lax}(C, D)$ and $CAl(C) \times_C \{X\}$.
(5) Bousfield localization of an E_∞-ring is still an E_∞-ring, this is directly by the fact $Sp^{\otimes} \rightleftarrows Sp_E^{\otimes}$ is a symmetric monoidal adjunction, which will induce an adjunction $CAl(Sp) \rightleftarrows CAl(Sp_E)$ by symmetric monoidal ∞-categorical machine. However, both model category and EKMM can not provide such a machine.
(6) If C is a 1-category, then $Sp(C) \simeq \{*\}$ is trivial. The stabilization for 1-category is meaningless.

Outlook and methodology

Preventing Russell's paradox

In order to consider the **category of all categories**, we need to add a set-theoretic axiom into ZFC, i.e. Grothendieck's Assumption:
$\forall$ cardinal κ, there exists an inaccessible cardinal $\tau > \kappa$. (A good reference: Chap 1, 代数学方法 1，李文威)

Methodology

By Grothendieck's Assumption,
1. When not involving **category of all categories**, technically we can treat all things as small. So all propositions not involving **category of all categories** will hold in any Grothendieck universe.
2. When involving **category of all categories**, for example $\mathcal{C}at_\infty$, we consider it as the ∞-category $\mathcal{C}at_\infty^\tau$ of all τ-small categories for an inaccessible cardinal τ. Choose a bigger inaccessible $\tau_2 > \tau$, then technically we can treat $\mathcal{C}at_\infty^\tau$ as a τ_2-small ∞-category in $\mathcal{C}at_\infty^{\tau_2}$.

Outlook and methodology

Preventing Russell's paradox

In order to consider the **category of all categories**, we need to add a set-theoretic axiom into ZFC, i.e. Grothendieck's Assumption:
$\forall$ cardinal κ, there exists an inaccessible cardinal $\tau > \kappa$. (A good reference: Chap 1, 代数学方法 1，李文威)

Methodology

By Grothendieck's Assumption,
1. When not involving **category of all categories**, technically we can treat all things as small. So all propositions not involving **category of all categories** will hold in any Grothendieck universe.
2. When involving **category of all categories**, for example $\mathcal{C}at_\infty$, we consider it as the ∞-category $\mathcal{C}at_\infty^\tau$ of all τ-small categories for an inaccessible cardinal τ. Choose a bigger inaccessible $\tau_2 > \tau$, then technically we can treat $\mathcal{C}at_\infty^\tau$ as a τ_2-small ∞-category in $\mathcal{C}at_\infty^{\tau_2}$.

Definition (Kan extension along a full subcategory)

Let $i : \mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, we say a functor $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension along i iff $\forall X \in \mathcal{C}$, $(\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ is a colimit diagram, i.e. $colim_{A \to X, A \in \mathcal{C}_0} F(A) \simeq F(X)$.

Theorem

The restriction $Fun^{LKan}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} Fun^{\exists LKan}(\mathcal{C}_0, \mathcal{D})$ is a categorical equivalence.

Example

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a category that admits small colimits, then
(1) A functor $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is a left Kan extension along the Yoneda embedding $i : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ iff F preserves small colimits.
(2) For any $f \in Fun(\mathcal{C}, \mathcal{D})$, there exists a left Kan extension $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ along i.
(3) And hence we have $Fun^{colim}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$ is an equivalence. (e.g. $sSet \to Top$)

Definition (Kan extension along a full subcategory)

Let $i : \mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, we say a functor $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension along i iff $\forall X \in \mathcal{C}$, $(\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ is a colimit diagram, i.e. $colim_{A \to X, A \in \mathcal{C}_0} F(A) \simeq F(X)$.

Theorem

The restriction $Fun^{LKan}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} Fun^{\exists LKan}(\mathcal{C}_0, \mathcal{D})$ *is a categorical equivalence.*

Example

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a category that admits small colimits, then
(1) A functor $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is a left Kan extension along the Yoneda embedding $i : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ iff F preserves small colimits.
(2) For any $f \in Fun(\mathcal{C}, \mathcal{D})$, there exists a left Kan extension $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ along i.
(3) And hence we have $Fun^{colim}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$ is an equivalence. (e.g. $sSet \to Top$)

Universal properties in the category of categories

Definition (Kan extension along a full subcategory)

Let $i : \mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, we say a functor $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension along i iff $\forall X \in \mathcal{C}$, $(\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ is a colimit diagram, i.e. $colim_{A \to X, A \in \mathcal{C}_0} F(A) \simeq F(X)$.

Theorem

The restriction $Fun^{LKan}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} Fun^{\exists LKan}(\mathcal{C}_0, \mathcal{D})$ *is a categorical equivalence.*

Example

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a category that admits small colimits, then
(1) A functor $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is a left Kan extension along the Yoneda embedding $i : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ iff F preserves small colimits.
(2) For any $f \in Fun(\mathcal{C}, \mathcal{D})$, there exists a left Kan extension $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ along i.
(3) And hence we have $Fun^{colim}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$ is an equivalence. (e.g. $sSet \to Top$)

Cocompletion

Definition

Let $\mathbb{K}$ be a collection of simplicial sets. We say that an ∞-category $\mathcal{C}$ is $\mathbb{K}$-cocomplete if it admits K-diagram colimits, for each $K \in \mathbb{K}$.
We say that a functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$ if the ∞-category $\widehat{\mathcal{C}}$ is $\mathbb{K}$-cocomplete and for every $\mathbb{K}$-cocomplete ∞-category $\mathcal{D}$, precomposition with h induces an equivalence of ∞-categories
$\mathrm{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Theorem

Let $\mathbb{K}$ be a (small) collection of simplicial sets, then for any (small) ∞-category C, there exists a $\mathbb{K}$-completion $C \to P^{\mathbb{K}}(C)$. That gives an adjunction $Cat_\infty \rightleftarrows Cat(\mathbb{K})_\infty$. e.g. $P^{small}(C) = Fun(C, \mathcal{S})$ and $P^{small}(*) = \mathcal{S}$.

Cocompletion

Definition

Let $\mathbb{K}$ be a collection of simplicial sets. We say that an ∞-category $\mathcal{C}$ is $\mathbb{K}$-cocomplete if it admits K-diagram colimits, for each $K \in \mathbb{K}$.
We say that a functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$ if the ∞-category $\widehat{\mathcal{C}}$ is $\mathbb{K}$-cocomplete and for every $\mathbb{K}$-cocomplete ∞-category $\mathcal{D}$, precomposition with h induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Theorem

Let $\mathbb{K}$ be a (small) collection of simplicial sets, then for any (small) ∞-category C, there exists a $\mathbb{K}$-completion $C \to P^{\mathbb{K}}(C)$. That gives an adjunction $Cat_\infty \rightleftarrows Cat(\mathbb{K})_\infty$. e.g. $P^{small}(C) = Fun(C, \mathcal{S})$ and $P^{small}(*) = \mathcal{S}$.

More examples of universal properties

Let $\mathcal{D}$ be an ∞-category.

Theorem (Pointedlization)

If $\mathcal{D}$ admits final object, then there exists a pointedlization $\mathcal{D}_{*/} \to \mathcal{D}$ such that for any pointed ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Final,pt} \rightleftarrows Cat_\infty^{Final}$.

Theorem (Stabilization)

If $\mathcal{D}$ admits finite limits, then there exists a stabilization $Sp(\mathcal{D}) \to \mathcal{D}$ such that for any stable ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Flim,st} \rightleftarrows Cat_\infty^{Flim}$.

Example

The category spectra $Sp(P(*))$ is the stabilization of the cocompletion of the trivial ∞-category.

More examples of universal properties

Let $\mathcal{D}$ be an ∞-category.

Theorem (Pointedlization)

If $\mathcal{D}$ admits final object, then there exists a pointedlization $\mathcal{D}_{*/} \to \mathcal{D}$ such that for any pointed ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Final,pt} \rightleftarrows Cat_\infty^{Final}$.

Theorem (Stabilization)

If $\mathcal{D}$ admits finite limits, then there exists a stabilization $Sp(\mathcal{D}) \to \mathcal{D}$ such that for any stable ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Flim,st} \rightleftarrows Cat_\infty^{Flim}$.

Example

The category spectra $Sp(P(*))$ is the stabilization of the cocompletion of the trivial ∞-category.

More examples of universal properties

Let $\mathcal{D}$ be an ∞-category.

Theorem (Pointedlization)

If $\mathcal{D}$ admits final object, then there exists a pointedlization $\mathcal{D}_{*/} \to \mathcal{D}$ such that for any pointed ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Final,pt} \rightleftarrows Cat_\infty^{Final}$.

Theorem (Stabilization)

If $\mathcal{D}$ admits finite limits, then there exists a stabilization $Sp(\mathcal{D}) \to \mathcal{D}$ such that for any stable ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Flim,st} \rightleftarrows Cat_\infty^{Flim}$.

Example

The category spectra $Sp(P(*))$ is the stabilization of the cocompletion of the trivial ∞-category.

More examples of universal properties

Definition

Let $n \geq -2$, an object Z in an ∞-category C is n-truncated if, for every object $Y \in C$, the space $Map_C(Y, Z)$ is n-truncated space.

Theorem (Truncation)

If C is a presentable ∞-category, then there exists an n-truncation functor $C \to \tau_{\leq n} C$. Suppose that $\mathcal{D}$ is a presentable that all objects are n-truncated, i.e. it's an $(n+1)$-category. Then composition with $\tau_{\leq n}$ induces an equivalence $s : \mathrm{Fun}^{\mathrm{L}}(\tau_{\leq n}\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$. That provides an adjunction $Pr^L \rightleftarrows Pr^L_{\leq(n+1)}$.

Example

(1) An space X in $\mathcal{S}$ is n-truncated iff all $\pi_i X$ vanish when $i > n$. Particularly $\mathcal{S}_{\leq 0} \simeq N(Set)$.

(2) An n-truncated object Cat_∞ is exactly an n-category. And all n-categories form an $(n+1)$-category $(Cat_\infty)_{\leq n}$.

More examples of universal properties

Definition

Let $n \geq -2$, an object Z in an ∞-category C is n-truncated if, for every object $Y \in C$, the space $Map_C(Y, Z)$ is n-truncated space.

Theorem (Truncation)

If C is a presentable ∞-category, then there exists an n-truncation functor $C \to \tau_{\leq n} C$. Suppose that $\mathcal{D}$ is a presentable that all objects are n-truncated, i.e. it's an $(n+1)$-category. Then composition with $\tau_{\leq n}$ induces an equivalence $s : \mathrm{Fun}^{\mathrm{L}}(\tau_{\leq n}\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$. That provides an adjunction $Pr^L \rightleftarrows Pr^L_{\leq(n+1)}$.

Example

(1) An space X in $\mathcal{S}$ is n-truncated iff all $\pi_i X$ vanish when $i > n$. Particularly $\mathcal{S}_{\leq 0} \simeq N(Set)$.
(2) An n-truncated object Cat_∞ is exactly an n-category. And all n-categories form an $(n+1)$-category $(Cat_\infty)_{\leq n}$.

Definition (Reformulate commutative monoid)

A commutative monoid in an ordinary category C which admits finite products is a functor $M : (Fin_*)_{\leq 3} \to C$ such that the canonical maps $M(\rho_i) : M(\langle n\rangle) \to M(\langle 1\rangle)$ exhibit $M(\langle n\rangle) \simeq \prod_{1\leq i\leq n} M(\langle 1\rangle)$ in the C for $0 \leq n \leq 3$.

Definition

Let C be an ∞-category with finite products, we define a commutative monoid as a a functor $M : N_*(Fin_*) \to C$ such that the canonical maps $M(\rho_i) : M(\langle n\rangle) \to M(\langle 1\rangle)$ exhibit $M(\langle n\rangle) \simeq \prod_{1\leq i\leq n} M(\langle 1\rangle)$ in the C for all $n \geq 0$.

Proposition

Let C be an n-category with finite products, then $Fun^{CM}(N_*(Fin_*), C) \xrightarrow{\sim} Fun^{CM}(N_*(Fin_*)_{\leq(n+2)}, C)$ is categorical equivalent since in this case any commutative monoid $M : N_*(Fin_*) \to C$ is a right Kan extension along $N_*(Fin_*)_{\leq(n+2)}$.

Higher commutative monoid

Definition (Reformulate commutative monoid)

A commutative monoid in an ordinary category C which admits finite products is a functor $M : (Fin_*)_{\leq 3} \to C$ such that the canonical maps $M(\rho_i) : M(\langle n \rangle) \to M(\langle 1 \rangle)$ exhibit $M(\langle n \rangle) \simeq \prod_{1 \leq i \leq n} M(\langle 1 \rangle)$ in the C for $0 \leq n \leq 3$.

Definition

Let C be an ∞-category with finite products, we define a commutative monoid as a a functor $M : N_*(Fin_*) \to C$ such that the canonical maps $M(\rho_i) : M(\langle n \rangle) \to M(\langle 1 \rangle)$ exhibit $M(\langle n \rangle) \simeq \prod_{1 \leq i \leq n} M(\langle 1 \rangle)$ in the C for all $n \geq 0$.

Proposition

Let C be an n-category with finite products, then $Fun^{CM}(N_*(Fin_*), C) \xrightarrow{\sim} Fun^{CM}(N_*(Fin_*)_{\leq (n+2)}, C)$ is categorical equivalent since in this case any commutative monoid $M : N_*(Fin_*) \to C$ is a right Kan extension along $N_*(Fin_*)_{\leq (n+2)}$.

Higher commutative monoid

Definition (Reformulate commutative monoid)

A commutative monoid in an ordinary category C which admits finite products is a functor $M : (Fin_*)_{\leq 3} \to C$ such that the canonical maps $M(\rho_i) : M(\langle n \rangle) \to M(\langle 1 \rangle)$ exhibit $M(\langle n \rangle) \simeq \prod_{1 \leq i \leq n} M(\langle 1 \rangle)$ in the C for $0 \leq n \leq 3$.

Definition

Let C be an ∞-category with finite products, we define a commutative monoid as a a functor $M : N_*(Fin_*) \to C$ such that the canonical maps $M(\rho_i) : M(\langle n \rangle) \to M(\langle 1 \rangle)$ exhibit $M(\langle n \rangle) \simeq \prod_{1 \leq i \leq n} M(\langle 1 \rangle)$ in the C for all $n \geq 0$.

Proposition

Let C be an n-category with finite products, then $Fun^{CM}(N_*(Fin_*), C) \xrightarrow{\sim} Fun^{CM}(N_*(Fin_*)_{\leq (n+2)}, C)$ is categorical equivalent since in this case any commutative monoid $M : N_*(Fin_*) \to C$ is a right Kan extension along $N_*(Fin_*)_{\leq (n+2)}$.

Symmetric monoidal ∞-category

Definition

A symmetric monoidal ∞-category is a commutative monoid in Cat_∞. Particularly, when a symmetric monoidal ∞-category C is 1-category, it is a commutative monoid in the $(Cat_\infty)_{\leq 1}$, which is a 2-category. So we have $CMon(Cat_{\leq 1}) \xrightarrow{\sim} Fun^{CM}(N_*(Fin_*)_{\leq 4}, (Cat_\infty)_{\leq 1})$.

By (un)straightening equivalence $Fun(N_*(Fin_*), Cat_\infty) \simeq CoCart_{/N_*(Fin_*)}$, we get the following equivalent definition.

Definition

A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : C^{\otimes} \to N_*(Fin_*)$ with the following property:
For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho^i_! : C^{\otimes}_{\langle n \rangle} \to C^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $C^{\otimes}_{\langle n \rangle} \simeq (C^{\otimes}_{\langle 1 \rangle})^n$. And define $C^{\otimes}_{\langle 1 \rangle}$ as its underlying ∞-category.

Definition

A symmetric monoidal ∞-category is a commutative monoid in Cat_∞. Particularly, when a symmetric monoidal ∞-category C is 1-category, it is a commutative monoid in the $(Cat_\infty)_{\leq 1}$, which is a 2-category. So we have $CMon(Cat_{\leq 1}) \xrightarrow{\sim} Fun^{CM}(N_*(Fin_*)_{\leq 4}, (Cat_\infty)_{\leq 1})$.

By (un)straightening equivalence $Fun(N_*(Fin_*), Cat_\infty) \simeq CoCart_{/N_*(Fin_*)}$, we get the following equivalent definition.

Definition

A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : \mathcal{C}^{\otimes} \to N_*(Fin_*)$ with the following property: For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$. And define $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ as its underlying ∞-category.

Tensor product of ∞-categories

Let $\mathbb{K}$ be the collection of all small simplicial sets.

Definition

Given 2 cocomplete ∞-categories C and D, we define the tensor product as a functor $C \times D \to C \otimes D$ such that for any cocomplete E, we have $Fun^{\mathbb{K}}(C \otimes D, E) \xrightarrow{\sim} Fun^{\mathbb{K}\boxtimes\mathbb{K}}(C \times D, E)$. Such tensor product always exists because the natural functor $C \times D \to \mathcal{P}^{\mathbb{K}}_{\mathbb{K}\boxtimes\mathbb{K}}(C \times D)$ satisfies that.

Theorem

The above gives a symmetric monoidal structure $\widehat{Cat}_\infty(\mathbb{K})^\otimes \to N_*(Fin_*)$ *and makes the cocompletion funcor a symmetric monoidal adjunction* $\widehat{Cat}_\infty^\otimes \rightleftarrows \widehat{Cat}_\infty(\mathbb{K})^\otimes$. *So* $S = \mathcal{P}(*)$ *is the unit in* $\widehat{Cat}_\infty(\mathbb{K})^\otimes$.

Tensor product of ∞-categories

Let $\mathbb{K}$ be the collection of all small simplicial sets.

Definition

Given 2 cocomplete ∞-categories C and D, we define the tensor product as a functor $C \times D \to C \otimes D$ such that for any cocomplete E, we have $Fun^{\mathbb{K}}(C \otimes D, E) \xrightarrow{\sim} Fun^{\mathbb{K}\boxtimes\mathbb{K}}(C \times D, E)$. Such tensor product always exists because the natural functor $C \times D \to \mathcal{P}^{\mathbb{K}}_{\mathbb{K}\boxtimes\mathbb{K}}(C \times D)$ satisfies that.

Theorem

The above gives a symmetric monoidal structure $\widehat{Cat}_{\infty}(\mathbb{K})^{\otimes} \to N_*(Fin_*)$ *and makes the cocompletion funcor a symmetric monoidal adjunction* $\widehat{Cat}^{\otimes}_{\infty} \rightleftarrows \widehat{Cat}_{\infty}(\mathbb{K})^{\otimes}$. *So* $\mathcal{S} = \mathcal{P}(*)$ *is the unit in* $\widehat{Cat}_{\infty}(\mathbb{K})^{\otimes}$.

Cocomplete symmetric monoidal structures

Remark

By (un)straightening equivalence, $CAl(\widehat{Cat}_\infty(\mathbb{K})) \subset CAl(\widehat{Cat}_\infty)$ is the subcategory whose objects are symmetric monoidal ∞-categories such that $- \otimes -$ preserves colimits separately in each variable (called **cocomplete symmetric monoidal** categories), and whose morphisms are **colimit-preserving** symmetric monoidal functors.

Corollary

The symmetric monoidal adjunction induces an adjunction between algebras $F : CAl(\widehat{Cat}_\infty) \rightleftarrows CAl(\widehat{Cat}_\infty(\mathbb{K}))$.

Corollary

(1) The $\mathcal{S} = \mathcal{P}(*)$ is the unit in $\widehat{Cat}_\infty(\mathbb{K})^\otimes$, which means it is initial object in $CAl(\widehat{Cat}_\infty(\mathbb{K}))$ and hence $\mathcal{S}$ admits a cocomplete symmetric monoidal structure $\mathcal{S}$.
(2) So for any cocomplete symmetric monoidal ∞-category, there exists essentially unique colimit-preserving symmetric monoidal functor $\mathcal{S}^\otimes \to \mathcal{C}^\otimes$.

Cocomplete symmetric monoidal structures

Remark

By (un)straightening equivalence, $CAl(\widehat{Cat}_\infty(\mathbb{K})) \subset CAl(\widehat{Cat}_\infty)$ is the subcategory whose objects are symmetric monoidal ∞-categories such that $- \otimes -$ preserves colimits separately in each variable (called **cocomplete symmetric monoidal** categories), and whose morphisms are **colimit-preserving** symmetric monoidal functors.

Corollary

The symmetric monoidal adjunction induces an adjunction between algebras $F : CAl(\widehat{Cat}_\infty) \rightleftarrows CAl(\widehat{Cat}_\infty(\mathbb{K}))$.

Corollary

(1) The $\mathcal{S} = \mathcal{P}(*)$ is the unit in $\widehat{Cat}_\infty(\mathbb{K})^\otimes$, which means it is initial object in $CAl(\widehat{Cat}_\infty(\mathbb{K}))$ and hence $\mathcal{S}$ admits a cocomplete symmetric monoidal structure $\mathcal{S}$.
(2) So for any cocomplete symmetric monoidal ∞-category, there exists essentially unique colimit-preserving symmetric monoidal functor $\mathcal{S}^\otimes \to \mathcal{C}^\otimes$.

Proposition (Localization)

Let $\mathcal{C}$ be an ∞-category and let $L : \mathcal{C} \to \mathcal{C}$ be a functor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. The following conditions are equivalent:
(1) There exists a functor $f : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $g : \mathcal{D} \to \mathcal{C}$ and an equivalence between $g \circ f$ and L.
(2) When regarded as a functor from $\mathcal{C}$ to $L\mathcal{C}$, L is a left adjoint of the inclusion $L\mathcal{C} \subseteq \mathcal{C}$.
(3) There exists a natural transformation from $\mathrm{id}_{\mathcal{C}} \to L$ such that, $L \circ \mathrm{id}_{\mathcal{C}} \to L \circ L$ and $\mathrm{id}_{\mathcal{C}} \circ L \to L \circ L$ are equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, i.e. an idempotent object in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

Proposition

The full subcat $Pr^L \subset \widehat{Cat}_\infty(\mathbb{K})$ is closed under tensor product and hence inherits a symmetric monoidal structure $Pr_L^{\otimes}$.

Bousfield localization

Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal category, i.e. an object in $CAl(Pr^L)$.

Theorem (Bousfield localization)

Let $E \in \mathcal{C}$ be an object, then $W_E = \{X \to Y | X \otimes E \xrightarrow{\sim} Y \otimes E\} \subset Fun(\Delta^1, \mathcal{C})$ is a small-generated strongly saturated collection, which means there exists an accessible localization functor $L_E : \mathcal{C} \to \mathcal{C}$.

Furthermore, Bousfield localization is compatible with its symmetric monoidal structure, meaning it forms a symmetric monoidal adjunction $\mathcal{C}^{\otimes} \rightleftarrows \mathcal{C}_E^{\otimes}$.

Corollary

Symmetric monoidal adjunction gives an adjunction $CAl(\mathcal{C}) \rightleftarrows CAl(\mathcal{C}_E)$. And a morphims $A \to B$ in $CAl(\mathcal{C})$ is a $CAl(\mathcal{C}_E)$-localization iff underlying $p(A) \to p(B)$ is an E-localization in $\mathcal{C}$.

Localization and idempotent object

Definition (idempotent object)

Let C be a monoidal (∞-)category. A morphism $1_C \to X$ is idempotent iff $1_C \otimes X \to X \otimes X$ and $X \otimes 1_C \to X \otimes X$ are equivalences. (e.g. $\mathbb{Z} \to \mathbb{Z}[1/p]$)

Theorem

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $e : \mathbf{1} \to E$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:

(1) The map e exhibits E as an idempotent object of $\mathcal{C}$.

(2) Let $l_E : \mathcal{C} \to \mathcal{C}$ be the functor given by left tensor product with E. Then e induces a functor $\alpha : \mathrm{id}_{\mathcal{C}} \to l_E$ which exhibits l_E as a localization functor on $\mathcal{C}$.

Localization and idempotent object

Definition (idempotent object)

Let C be a monoidal (∞-)category. A morphism $1_C \to X$ is idempotent iff $1_C \otimes X \to X \otimes X$ and $X \otimes 1_C \to X \otimes X$ are equivalences. (e.g. $\mathbb{Z} \to \mathbb{Z}[1/p]$)

Theorem

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $e : \mathbf{1} \to E$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:
(1) The map e exhibits E as an idempotent object of $\mathcal{C}$.
(2) Let $l_E : \mathcal{C} \to \mathcal{C}$ be the functor given by left tensor product with E. Then e induces a functor $\alpha : \mathrm{id}_{\mathcal{C}} \to l_E$ which exhibits l_E as a localization functor on $\mathcal{C}$.

Symmetric localization and idempotent algebra

Definition

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We will say that a commutative algebra object $A \in \mathrm{CAlg}(\mathrm{C})$ is idempotent if unit map $e : \mathbf{1} \to A$ is idempotent.

Theorem

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$, which we regard as a trivial algebra object of $\mathcal{C}$. Then the functor

$$\theta : \mathrm{CAlg}^{idem}(\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})_{\mathbf{1}/} \to \mathcal{C}_{\mathbf{1}/}$$

is fully faithful, and its essential image are idempotent objects in $\mathcal{C}$, which gives an equivalence $\mathrm{CAlg}^{idem}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}_{\mathbf{1}/})^{idem}$. Furthermore, any mapping space in $(\mathcal{C}_{\mathbf{1}/})^{idem}$ is either empty or contractible, i.e. it is a 0-category and equivalent to a partial-order set $N(I)$.

Symmetric localization and idempotent algebra

Definition

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We will say that a commutative algebra object $A \in \mathrm{CAlg}(\mathrm{C})$ is idempotent if unit map $e : \mathbf{1} \to A$ is idempotent.

Theorem

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$, which we regard as a trivial algebra object of $\mathcal{C}$. Then the functor

$$\theta : \mathrm{CAlg}^{idem}(\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})_{1/} \to \mathcal{C}_{\mathbf{1}/}$$

is fully faithful, and its essential image are idempotent objects in $\mathcal{C}$, which gives an equivalence $\mathrm{CAlg}^{idem}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}_{\mathbf{1}/})^{idem}$. Furthermore, any mapping space in $(\mathcal{C}_{\mathbf{1}/})^{idem}$ is either empty or contractible, i.e. it is a 0-category and equivalent to a partial-order set $N(I)$.

Interesting applications

Proposition

The full subcat $Pr^L \subset \widehat{Cat}_\infty(\mathbb{K})$ is closed under tensor product (so $\mathcal{S}$ is also the unit in Pr^L) and hence inherits a symmetric monoidal structure. In this case, for any $C, D \in Pr^L$, we have natural equivalence $C \otimes D \simeq RFun(C^{op}, D)$.

Theorem

The following 3 colimit-preserving functors $\mathcal{S} \xrightarrow{\tau_{\leq n}} \tau_{\leq n}\mathcal{S}$, $\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_*$ and $\mathcal{S} \xrightarrow{\Sigma^\infty_+} Sp$ are idempotent objects in Pr^L. Let $\mathcal{C}$ be a presentable ∞-category, then
(1) The functor $\tau_{\leq n}$ induces a map $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \tau_{\leq n}\mathcal{S} \simeq \tau_{\leq n}\mathcal{C}$ which exhibits θ as an n-truncation functor.
(2) The functor $(-)_+$ induces a map $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \mathcal{S}_* \simeq \mathcal{C}_*$ which exhibits θ as a copointedlization functor.
(3) The functor Σ^∞_+ induces a map $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes Sp \simeq Sp(\mathcal{C})$ which exhibits θ as a cospectralization functor.

Interesting applications

Proposition

The full subcat $Pr^L \subset \widehat{Cat}_\infty(\mathbb{K})$ *is closed under tensor product (so* $\mathcal{S}$ *is also the unit in* Pr^L*) and hence inherits a symmetric monoidal structure. In this case, for any* $C, D \in Pr^L$, *we have natural equivalence* $C \otimes D \simeq RFun(C^{op}, D)$.

Theorem

The following 3 colimit-preserving functors $\mathcal{S} \xrightarrow{\tau_{\leq n}} \tau_{\leq n}\mathcal{S}$, $\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_*$ *and* $\mathcal{S} \xrightarrow{\Sigma^\infty_+} Sp$ *are idempotent objects in* Pr^L. *Let* $\mathcal{C}$ *be a presentable* ∞*-category, then*
(1) The functor $\tau_{\leq n}$ *induces a map* $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \tau_{\leq n}\mathcal{S} \simeq \tau_{\leq n}\mathcal{C}$ *which exhibits* θ *as an* n*-truncation functor.*
(2) The functor $(-)_+$ *induces a map* $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \mathcal{S}_* \simeq \mathcal{C}_*$ *which exhibits* θ *as a copointedlization functor.*
(3) The functor Σ^∞_+ *induces a map* $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes Sp \simeq Sp(\mathcal{C})$ *which exhibits* θ *as a cospectralization functor.*

Reinterpretation

Corollary

By $\mathrm{CAlg}(Pr^L)^{\text{idem}} \xrightarrow{\sim} (Pr^L_{\mathcal{S}/})^{\text{idem}}$ and the fact that $\mathcal{S} \xrightarrow{\tau_{\leq n}} \tau_{\leq n}\mathcal{S}$, $\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_*$, $\mathcal{S} \xrightarrow{\Sigma^\infty_+} Sp \in (Pr^L_{\mathcal{S}/})^{\text{idem}}$,

(1) There is a unique cocomplete symmetric monoidal structure on $\mathcal{S}$ such that $*$ is the unit, which coincides its **Cartesian monoidal** structure.

(2) There is a unique cocomplete symmetric monoidal structure on $\tau_{\leq n}\mathcal{S}$ such that $*$ is the unit, which coincides its **Cartesian monoidal** structure.

(3) There is a unique cocomplete symmetric monoidal structure on $\mathcal{S}_*$ such that S^0 is the unit.

(4) There is a unique cocomplete symmetric monoidal structure on Sp such that $\Sigma^\infty S^0$ is the unit.

Bousfield localization with an idempotent object

Theorem

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $1_{\mathcal{C}} \to E$ be an idempotent object in $\mathcal{C}$, then there exists a symmetric monoidal localization $L_E^{\otimes} : \mathcal{C}^{\otimes} \rightleftarrows \mathcal{C}_E^{\otimes}$. Furthermore, The inclusion $\mathcal{C}_E^{\otimes} \to \mathcal{C}^{\otimes}$ is closed under tensor product and hence (strong) symmetric monoidal.

Corollary

The following 3 collections of presentable ∞-categories are closed under tensor product:
(1) Pointed presentable ∞-categories;
(2) Stable presentable ∞-categories;
(3) Presentable $(n+1)$-categories.