Motivic homotopy theory and calculation methods: On “The special fiber of the motivic deformation of the stable homotopy category is algebraic”

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Algebraic cobordism theory

For a fixed prime p, Voevodsky constructed the mod-p motivic Eilenberg-Mac Lane spectrum that represents the mod-p motivic cohomology. We denote it by $H\mathbb{F}_p^{mot}$. Its value at a point is $\pi_{*,*}H\mathbb{F}_p^{mot} = \mathbb{F}[\tau]$, and τ has bi-degree $(0,-1)$.

We denote by $S^{0,0}$ the motivic sphere spectrum. For the grading, we denote by $S^{1,0}$ the suspension spectrum of the simplicial sphere S^1 , and by $S^{1,1}$ the suspension spectrum of the multiplicative group $G_m = \mathbb{A}^1 \setminus \{0\}$.

The class τ can be lifted to a map between $H\mathbb{F}_p^{mot}$-completed motivic sphere spectra $\tau : \widehat{S}^{0,-1} \to \widehat{S}^{0,0}$ that induces a non-zero map on mod-p motivic homology. We denote by $\widehat{S}^{0,0}/\tau$ the cofiber of τ.

There is a Betti Realization functor Re from the motivic stable homotopy category over $\mathbb{C}$ to the classical stable homotopy category. $Re(S^{n,w}) \simeq S^{n,0}$ and $Re(H\mathbb{F}_p^{mot}) \simeq H\mathbb{F}_p$.

Algebraic cobordism theory

Let MGL be the cellular motivic algebraic cobordism spectrum introduced by Voevodsky.

We define

$$MU^{mot} := MGL \wedge_{S^{0,0}} \widehat{S}^{0,0}.$$

The motivic homotopy groups are computed by Hu–Kriz-Ormsby and Dugger–Isaksen:

$$\pi_{*,*}(MU^{mot}) = \mathbb{Z}_p[\tau][x_1, x_2, \cdots] \ , \ deg(\tau) = (0, -1) \ , \ def(x_i) = (2i, i)$$

The spectrum $MU^{mot}/\tau := \widehat{S}^{0,0}/\tau \wedge_{\widehat{S}^{0,0}} MU^{mot}$ has motivic homotopy groups:

$$\pi_{*,*}(MU^{mot}/\tau) = \mathbb{Z}_p[x_1, x_2, \cdots] = MU^{mot}_{*,*}/\tau$$

t-structure

Definition 2.1. A *t-structure* on a stable ∞-category $\mathcal{C}$ is a pair of two full subcategories $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}$ that are stable under equivalences, satisfying the following three properties:

(1) for $X \in \mathcal{C}_{\geqslant 0}$ and $Y \in \Sigma^{-1}\mathcal{C}_{\leqslant 0}$, we have $[X, Y]_{\mathcal{C}} = 0$;

(2) there are inclusions $\Sigma\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}_{\geqslant 0}$ and $\Sigma^{-1}\mathcal{C}_{\leqslant 0} \subseteq \mathcal{C}_{\leqslant 0}$;

(3) for any $X \in \mathcal{C}$, there exists a fiber sequence

$$X_{\geqslant 0} \longrightarrow X \longrightarrow X_{\leqslant -1},$$

with $X_{\geqslant 0} \in \mathcal{C}_{\geqslant 0}$ and $X_{\leqslant -1} \in \Sigma^{-1}\mathcal{C}_{\leqslant 0}$.

t-exact and bounded

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. We say that an exact functor $f\colon\mathcal{C}\to\mathcal{C}'$ is *right t-exact*, if it carries $\mathcal{C}_{\geqslant 0}$ to $\mathcal{C}'_{\geqslant 0}$. An exact functor $f\colon\mathcal{C}\to\mathcal{C}'$ is *left t-exact*, if it carries $\mathcal{C}_{\leqslant 0}$ to $\mathcal{C}'_{\leqslant 0}$. A functor is *$t$-exact* if it is both left and right t-exact.

Definition 2.4. Denote by $\mathcal{C}^{+}$ and $\mathcal{C}^{-}$ the stable full subcategories spanned by *left-bounded* and *right-bounded* objects in $\mathcal{C}$, respectively:

$$\mathcal{C}^{+}=\bigcup_{n\geqslant 0}\mathcal{C}_{\leqslant n} \quad \text{and} \quad \mathcal{C}^{-}=\bigcup_{n\geqslant 0}\mathcal{C}_{\geqslant -n},$$

and by

$$\mathcal{C}^{\mathrm{b}}:=\mathcal{C}^{+}\cap\mathcal{C}^{-}$$

the stable subcategory of *bounded objects*. We say that the t-structure is *left-bounded*,

Heart of a t-structure

right-bounded or *bounded*, if the inclusion of $\mathcal{C}^{+}$, $\mathcal{C}^{-}$ or $\mathcal{C}^{\mathrm{b}}$, respectively, in $\mathcal{C}$, is an equivalence.

The intersection

$$\mathcal{C}^{\heartsuit}=\mathcal{C}_{\geqslant 0}\cap\mathcal{C}_{\leqslant 0}$$

is called the *heart* of the t-structure.

The ∞-category $\mathcal{C}^{\heartsuit}$ is always equivalent to (the nerve of) its homotopy category $h\mathcal{C}^{\heartsuit}$, which is an abelian category (see [41, Remark 1.2.1.12]). Following [41], we abuse the notation by identifying $\mathcal{C}^{\heartsuit}$ with the abelian category $h\mathcal{C}^{\heartsuit}$.

Chow–Novikov degree

Definition

For any motivic spectrum X, consider its bigraded motivic homotopy groups $\pi_{s,w}X$. Here, s is the topological degree under the Betti realization, and w is the motivic weight. The **Chow–Novikov degree** of an element in $\pi_{s,w}X$ is defined as $s - 2w$. We say that $\pi_{s,w}X$ is concentrated in Chow–Novikov degrees I, where I is a set of integers, if all non-zero elements in $\pi_{*,*}X$ are concentrated in Chow–Novikov degrees belonging to I.

For example, the homotopy groups of MU^{mot}/τ are concentrated in Chow–Novikov degree zero, while the homotopy groups of MU^{mot} are concentrated in non-negative even Chow–Novikov degrees.

$$\pi_{*,*}(MU^{mot}) = \mathbb{Z}_p[\tau][x_1, x_2, \cdots] \ , \ deg(\tau) = (0, -1) \ , \ def(x_i) = (2i, i)$$

$$\pi_{*,*}(MU^{mot}/\tau) = \mathbb{Z}_p[x_1, x_2, \cdots] = MU^{mot}_{*,*}/\tau$$

Chow t-structure

- We define $MU^{mot}/\tau\text{-}Mod^{b}_{cell}$ as the stable full subcategory of $MU^{mot}/\tau\text{-}Mod_{cell}$ spanned by objects whose homotopy groups are concentrated in bounded Chow-Novikov degrees.
- Define $MU^{mot}/\tau\text{-}Mod^{b,\geq 0}_{cell}$, $MU^{mot}/\tau\text{-}Mod^{b,\leq 0}_{cell}$, $MU^{mot}/\tau\text{-}Mod^{b,\heartsuit}_{cell}$ as the full subcategories of $MU^{mot}/\tau\text{-}Mod^{b}_{cell}$ cell spanned by objects whose homotopy groups are concentrated in non-negative, non-positive and zero Chow–Novikov degrees, respectively.
- Define $\widehat{S^{0,0}}/\tau\text{-}Mod^{b}_{harm}$ as the stable full subcategory of $\widehat{S^{0,0}}/\tau\text{-}Mod_{harm}$ spanned by objects whose MU^{mot} -homology groups are concentrated in bounded Chow–Novikov degrees.
- Define $\widehat{S^{0,0}}/\tau\text{-}Mod^{b,\geq}_{harm}$, $\widehat{S^{0,0}}/\tau\text{-}Mod^{b,\leq}_{harm}$, $\widehat{S^{0,0}}/\tau\text{-}Mod^{b,\heartsuit}_{harm}$ as the full subcategories of $\widehat{S^{0,0}}/\tau\text{-}Mod^{b}_{harm}$ spanned by objects whose MU^{mot}-homology groups are concentrated in non-negative, non-positive and zero Chow-Novikov degrees respectively.

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Main theorems 1

Theorem

There is an equivalence of stable ∞-categories equipped with t-structures at each prime p:

$$\mathcal{D}^b(BP_*BP\text{-}Comod^{ev}) \simeq \widehat{S^{0,0}}/\tau\text{-}Mod^b_{harm} \tag{1}$$

between the bounded derived category of p-completed BP_*BP-comodules that are concentrated in even degrees, and the category of harmonic motivic left module spectra over $\widehat{S^{0,0}}/\tau$, whose MGL-homology has bounded Chow-Novikov degree, with morphisms the $\widehat{S^{0,0}}/\tau$-linear map.

Here,$S^{0,0}/\tau$ is a motivic E_∞-ring spectrum, which is also known as the cofiber of τ.
The motivic spectrum MGL is the algebraic cobordism spectrum.
A motivic left-module spectrum over $S^{0,0}/\tau$ is harmonic, if it is $S^{0,0}/\tau$ -cellular and the map to its MGL-completion induces an isomorphism on $\pi_{*,*}$.

Main theorems 2

THEOREM 1.11. (1) *The full subcategories*

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geqslant 0} \quad \textit{and} \quad \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leqslant 0}$$

define a t-structure on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}$.

(2) *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit} \longrightarrow \mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}$$

is an equivalence.

(3) *There exists an equivalence of stable ∞-categories*

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}),$$

that preserves the given t-structures and extends the functor $\pi_{*,*}$ on the heart.

Main theorems 3

THEOREM 1.13. (1) *The full subcategories*

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}} \quad \textit{and} \quad \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$$

define a t*-structure on* $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$.

(2) *The functor*

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\colon \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} \longrightarrow \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}$$

is an equivalence.

(3) *There exists an equivalence of stable* ∞*-categories*

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}})$$

that preserves the given t*-structures and extends the functor* $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ *on the heart.*

Lurie's theorem

PROPOSITION 2.12. *Let $\mathcal{C}$ be a stable ∞-category with a given bounded t-structure. Suppose that the following conditions hold:*

(1) *the abelian category $\mathcal{A}{=}h\mathcal{C}^{\heartsuit}$ has enough injective objects;*

(2) *for any pair of objects $X, Y \in \mathcal{A}$, if Y is injective, then the abelian groups*

$$[\Sigma^{-i}X, Y]_{\mathcal{C}}$$

vanish for $i{>}0$.

Then, there exists an equivalence of stable ∞-categories

$$G\colon \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $N(\mathcal{A}){\simeq}\mathcal{C}^{\heartsuit}{\subseteq}\mathcal{C}$, and which preserves t-structures. Here, $N(\mathcal{A})$ is the nerve of the abelian category $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$ is the bounded derived category of $\mathcal{A}$.

Uniersal coefficient SS thm

THEOREM 3.2. (Universal coefficient spectral sequence) *For any*

$$X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}},$$

there is a conditionally convergent spectral sequence

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

Moreover, if both $\pi_{*,*}X$ *and* $\pi_{*,*}Y$ *are concentrated in bounded Chow–Novikov degrees, then the spectral sequence converges strongly and collapses at a finite page.*

Proof: For degree reasons.

Isomorphism between topological and algebraic Hom set

Corollary 3.3. *Let*

$$X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}} \quad \textit{and} \quad Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}}.$$

The abelian group of homotopy classes of bi-degree $(0,0)$ *can be computed algebraically by the isomorphism*

$$[X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}{}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y)$$

that is induced by applying $\pi_{*,*}$.

Proof: For degree reasons.

The equivalence on the heart

Corollary 3.4. *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit} \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$$

is fully faithful. Here, the right-hand side is understood as a discrete ∞*-category.*

As a consequence, Corollary 3.4 shows that $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$ is also a discrete ∞-category.

Proof. For $n\geqslant 0$ and two objects $X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$, by Corollary 3.3, the edge homomorphism

$$[\Sigma^{n,0}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \xrightarrow{\pi_{*,*}} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*}/\tau}(\pi_{*,*}\Sigma^{n,0}X, \pi_{*,*}Y)$$

is an isomorphism. When $n>0$, the bigraded module $\pi_{*,*}\Sigma^{n,0}X$ is concentrated in positive Chow–Novikov degree. So the right-hand side of the above isomorphism is concentrated in the case $n=0$. This shows that $\pi_{*,*}$ is fully faithful on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$. □

The equivalence on the heart 2

To show the equivalence on the heart, we only need to show the **essential surjectivity** of $\pi_{*,*}$. We need to show that any object $M \in MU^{mot}_{*,*}/\tau\text{-}mod^0$ can be realized as the homotopy groups of an object in $MU^{mot}/\tau\text{-}mod^{heart}_{cell}$.

The free $MU^{mot}_{*,*}/\tau$-module can be realized by wedge of spectra MU^{mot}/τ.For an arbitrary M, we can pick a free resolution $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$, and they can be realized as $Z_0 \leftarrow Z_1 \leftarrow \cdots$. and we can construct a tower $X_1 \rightarrow X_2 \rightarrow \cdots$ s.t. $\Sigma^{i,0} Z_{i+1} \rightarrow X_i \rightarrow X_{i+1}$ is a cofiber sequence and $X := colim(X_1 \rightarrow X_2 \rightarrow \cdots)$, $\pi_{*,*}(X) = M$.

Proposition

The functor

$$\pi_{*,*} : MU^{mot}/\tau\text{-}Mod^{\heartsuit}_{cell} \rightarrow MU^{mot}_{*,*}/\tau\text{-}Mod^0$$

is an equivalence of ∞-categories.

The absolute Adams–Novikov SS

For the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{harm}}$, the t-structure is defined in terms of $\mathrm{MU}^{\mathrm{mot}}$-homology. We therefore need a version of the motivic Adams–Novikov spectral sequence that computes $\widehat{S^{0,0}}/\tau$-linear maps.

Recall from Dugger–Isaksen [14, §8] or Hu–Kriz–Ormsby [28] the usual $\mathrm{MU}^{\mathrm{mot}}$-based motivic Adams–Novikov spectral sequence

$$\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}}(\mathrm{MU}^{\mathrm{mot}}_{*,*}\widehat{S^{0,0}}, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow \pi_{*,*}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}.$$

This spectral sequence is not what we need. We need a spectral sequence of the form

$$\mathrm{Ext}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau},$$

Modified strategy

To show the equivalence on the heart, we only need to show the essential surjectivity of $\mathrm{MU}^{\mathrm{mot}}_{*,*}$.

Unlike the case for modules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, we do not have free resolutions for comodules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$. We will instead use Landweber's filtration theorem to realize all comodules that are finitely presented, and then extend the result using filtered colimits. In particular, all Smith–Toda complexes exist in $\widehat{S^{0,0}}/\tau$-**Mod**.

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Comparison of ASS and ANSS

The Adams spectral sequence and the Adams–Novikov spectral sequence are two of the most effective methods of computing the homotopy groups of the p-completed sphere spectrum of the form:

$$Ext_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \cong E_2^{s,t} \Rightarrow \pi_{t-s}\mathbb{S}^0$$
$$Ext_{BP_*BP}^{s,t}(BP_*, BP_*) \cong E_2^{s,t} \Rightarrow \pi_{t-s}\mathbb{S}^0$$

It is important to understand connections between them. A first connection is given by the Thom reduction map $\rho : BP \to H\mathbb{F}_p$, which is a ring spectra map and its behavior on the coefficient ring is given by $\rho_*(v_n) = 0$ for all $v_n \in BP_*(pt)$ and induces a map of spectral sequences:

$$Ext_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \to Ext_{BP_*BP}^{s,t}(BP_*, BP_*)$$

that preserves the (s, t)-degrees. However, a general homotopy class in $\pi_*\mathbb{S}^0$ usually have different Adams filtration and Adams-Novikov filtration. So this map is not very useful for comparison of the Adams filtration and the Adams-Novikov filtration of a surviving homotopy class, it only tells us the latter is less or equal to the former.

algebraic Novikov SS and Cartan-Eilenberg SS

A fundamental connection is the Miller square. We have an algebraic Novikov spectral sequence converging to the Adams-Novikov E^2-page, and a Cartan-Eilenberg spectral sequence converging to the Adams E^2-page. It turns out the E^2-pages of these two algebraic spectral sequences are isomorphic.
The algebraic Novikov spectral equence comes the filtration of powers of the augmentation ideal $I = (p, v_1, v_2, \cdots) \subset BP_*$,

$$E_2^{s,k,t} \cong Ext^{s,t}_{BP_*BP/I}(BP_*/I, I^k/I^{k+1}) \Rightarrow Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$$

The Cartan-Eilenberg spectral sequence is:

$$Ext^{s,t}_P(\mathbb{F}_p, Ext^k_Q(\mathbb{F}_p, \mathbb{F}_p)) \Rightarrow Ext^{s+k,t}_A(\mathbb{F}_p, \mathbb{F}_p)$$

where P is a sub-Hopf algebra of A and $Q = A \otimes_P \mathbb{F}_2$, We identify the E2-pages of the Cartan-Eilenberg spectral sequence and the algebraic Novikov spectral sequence by using the isomorphism of Hopf algebroids $(BP_*/I, BP_*BP/I) \cong (\mathbb{F}_p, P)$.

Miller square

So there is an isomorphism of E_2 page:

$$Ext_P^{s,t}(\mathbb{F}_p, Ext_Q^k(\mathbb{F}_p, \mathbb{F}_p)) \xrightarrow{\cong} Ext^{s,t'}_{BP_*BP/I}(BP_*/I, I^k/I^{k+1})$$

$$\begin{tikzcd}
 & & \mathrm{Ext}^{s,t}_{P_*}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[dl, Rightarrow, no head] \arrow[dr, Rightarrow, no head] & & \\
 & \text{Cartan–Eilenberg SS} \arrow[dl, Rightarrow] & & \text{Algebraic Novikov SS} \arrow[dr, Rightarrow] & \\
\mathrm{Ext}^{a,t+a-s}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[dr, Rightarrow, no head] & & & & \mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[dl, Rightarrow, no head] \\
 & \text{Adams SS} \arrow[dr, Rightarrow] & & \text{Adams–Novikov SS} \arrow[dl, Rightarrow] & \\
 & & \pi_{t-s}\widehat{S^0}. & &
\end{tikzcd}$$

Equivalence of spectral sequences

THEOREM 8.3. *At each prime p, there is an isomorphism of tri-graded spectral sequences between the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, which converges to the motivic homotopy groups of $\widehat{S^{0,0}}/\tau$, and the regraded algebraic Novikov spectral sequence, which converges to the Adams–Novikov E_2-page for the sphere spectrum.*

The indexes are indicated in the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^{a-s}/I^{a-s+1}) \arrow[r, "\cong"] \arrow[d, Rightarrow, "\text{Algebraic Novikov SS}"'] & \mathrm{Ext}^{a,2w-s+a,w}_{A^{\mathrm{mot}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "\text{Motivic Adams SS}"] \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[r, "\cong"] & \pi_{2w-s,w}(\widehat{S^{0,0}}/\tau).
\end{tikzcd}$$

Here, $A^{\mathrm{mot}}_{*,*}$ is the motivic mod-p *dual Steenrod algebra.*

Miller square and motivic SS 1

Which spectral sequence can we put in between these two spectral sequences and have a zig-zag diagram? Namely,

$$\begin{tikzcd}
\mathrm{Ext}_{P_*}^{s,t}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[d, Rightarrow, "\text{Algebraic Novikov SS}" description] & ? \arrow[l] \arrow[r] \arrow[d, Rightarrow] & \mathrm{Ext}_{A_*}^{a,t}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "\text{Adams SS}" description] \\
\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}^{s,t}(\mathrm{BP}_*, \mathrm{BP}_*) & ? \arrow[l] \arrow[r] & \pi_* \widehat{S}^0.
\end{tikzcd}$$

The answer is in the motivic world! It has been given by the equivalence of spectral sequences between Algebraic Novikov and motivic Adams of $\widehat{S^{0,0}}/\tau$.

Miller square and motivic SS 2

Combine the motivic deformation and the naturality of the motivic Adams spectral sequences and the equivalence of spectral sequences give us a zig-zag diagram.

$$\begin{tikzcd}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[r, "\cong"] \arrow[d, Rightarrow, "\text{Algebraic Novikov SS}" description] & \mathrm{Ext}^{s,t+a-s,t/2}_{A^{\mathrm{mot}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "\text{Motivic Adams SS}" description] & \mathrm{Ext}^{s,t+a-s,t/2}_{A^{\mathrm{mot}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \arrow[l] \arrow[r] \arrow[d, Rightarrow, "\text{Motivic Adams SS}" description] & \mathrm{Ext}^{a,t+a-s}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "\text{Adams SS}" description] \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[r, "\cong"] & \pi_{*,*}\widehat{S^{0,0}}/\tau & \pi_{*,*}\widehat{S^{0,0}} \arrow[l] \arrow[r, "\mathbf{Re}"] & \pi_*\widehat{S^0}.
\end{tikzcd}$$

Computation strategy I

Isaksen, Wang and Xu extend the computation of classical and motivic stable stems into a large range using the following steps:

1. Use a computer to carry out the entirely algebraic computation of the cohomology of the $\mathbb{C}$-motivic Steenrod algebra. These groups serve as the input to the $\mathbb{C}$-motivic Adams spectral sequence.
2. Use a computer to carry out the entirely algebraic computation of the algebraic Novikov spectral sequence that converges to the cohomology of the Hopf algebroid (BP_*, BP_*BP). This includes all differentials, and the multiplicative structure of the cohomology of (BP_*, BP_*BP).
3. Identify the algebraic Novikov spectral sequence with the motivic Adams spectral sequence that computes the homotopy groups of $\widehat{S^{0,0}/\tau}$. This includes an identification of $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ and $\pi_{*,*}\widehat{S}^{0,0}/\tau$.
4. Use the inclusion of the bottom cell and the projection to the top cell to pull back and push forward Adams differentials for $\widehat{S}^{0,0}/\tau$ to Adams differentials for the motivic sphere $\widehat{S}^{0,0}$.

Computation strategy II

5. Apply a variety of ad-hoc arguments to deduce additional Adams differentials for the motivic sphere. The most important method involves shuffling Toda brackets.
6. Use a long exact sequence in homotopy groups to deduce hidden τ-extensions in the motivic Adams spectral sequence for the sphere.
7. Invert τ to obtain the classical Adams spectral sequence and the classical stable homotopy groups.

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Thanks!