The motivic Adams spectral sequence.

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Outline

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The main result

For motivic homotopy theory, we have an Adams-like spectral sequence which E_2 page is:

$$E_2^{s,t,u} = Ext_A^{s,(t+s,u)}(\mathbb{M}_2, \mathbb{M}_2), \quad (1)$$

and $d_r : E_r^{s,t,u} \to E_r^{s+r,t-1,u}$ which convergents to $\pi_{*,*}(S_H^\wedge)$.

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- $\mathbb{M}_2$: the bigraded motivic cohomology ring of $Spec(\mathbb{C})$.
- H is the mod 2 motivic Eilenberg-MacLane spectrum.

Outline

The motivic Steenrod algebra

Voevodsky showed the structure of $\mathbb{M}_2$ and the motivic version Adem relation.

Theorem (Voevodsky)

The bigraded ring $\mathbb{M}_2$ *is the polynomial ring* $\mathbb{F}_2[\tau]$ *on one generator* τ *of bidegree* $(0, 1)$.

The motivic Steenrod algebra

Voevodsky showed the structure of $\mathbb{M}_2$ and the motivic version Adem relation.

Theorem (Voevodsky)

The motivic Steenrod algebra A is the $\mathbb{M}_2$-algebra generated by elements Sq^{2k} and Sq^{2k-1} for all $k \geq 1$, of bidegrees $(2k, k)$ and $(2k-1, k-1)$ respectively, and satisfying the following relations for $a < 2b$:

$$Sq^a Sq^b = \sum_c \binom{b-1-c}{a-2c} \tau^? Sq^{a+b-c} Sq^c. \tag{2}$$

Where the τ has a bidegree of $(0, 1)$.

Relation with classical Adams spectral sequence

Idea: Remove τ to degenerate the motivic case into the classical one.

Definition

For any motivic spectrum X, let

$$\theta_X : H^{*,*}(X) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \to H^p(X(\mathbb{C})) \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}] \quad (3)$$

be the $\mathbb{M}_2[\tau^{-1}]$-linear map that takes a class α of weight w in $H^{*,*}(X)$ to $\tau^w \alpha(\mathbb{C})$.

Outline

Calculation of the Ext group.

$$\begin{array}{ccccccccc}
X & \xleftarrow{=} & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
 & & K_0 & & K_1 & & K_2 & &
\end{array}$$

Calculation of the Ext group.

We can get an Adams-like spectral sequence by applying $\pi_{*,u}$ on this resolution.

$$E_2^{s,t,u} = Ext_A^{s,(t+s,u)}(\mathbb{M}_2, \mathbb{M}_2), \quad (4)$$

and $d_r : E_r^{s,t,u} \to E_r^{s+r,t-1,u}$. This spectral sequence convergents to $\pi_{*,*}(S_H^{\wedge})$.

Calculation of the Ext group.

The free part of $Ext_A^{*,*}(\mathbb{M}_2, \mathbb{M}_2)$ is isomorphic to the free part of $Ext_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Theorem

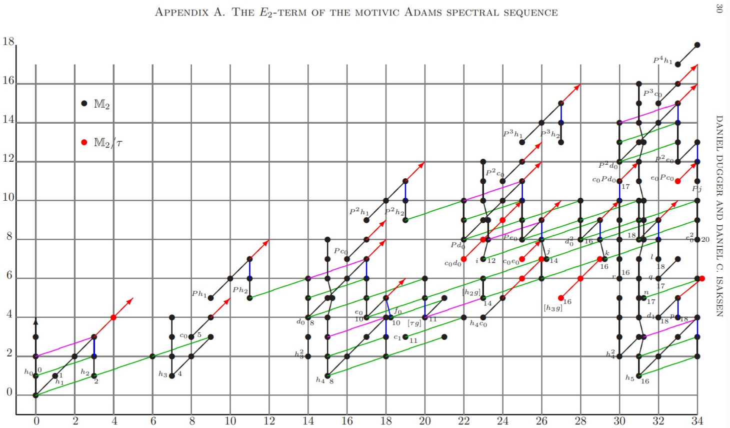
There is an isomorphism of rings

$$Ext_A^{*,*}(\mathbb{M}_2, \mathbb{M}_2) \otimes_{\tilde{\mathbb{M}}_2} \tilde{\mathbb{M}}_2[\tau^{-1}] \cong Ext_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}]. \quad (5)$$

Here

$$\tilde{\mathbb{M}}_2 := \mathbb{F}_2[\tilde{\tau}] = Hom_A^*(\mathbb{M}_2, \mathbb{M}_2) = Ext_A^{0,*}(\mathbb{M}_2, \mathbb{M}_2). \quad (6)$$

Calculation of the Ext group.



Calculation of the Ext group.

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- $Sq^2\beta_2 + \tau(Sq^1\beta_3 + Sq^4\beta_1) = 0$
- $\tau(Sq^3Sq^1\beta_2 + Sq^2Sq^1\beta_3 + (Sq^6 + \tau Sq^5Sq^1)\beta_1)$

Some other topic about computation

- Product

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- Differential

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- Ingredient: The Kunneth formula.

The universal coefficient theorem

θ_H is the canonical map of $H_{*,*}(H) \to Hom_{\mathbb{M}_2}(H^{*,*}(H), \mathbb{M}_2)$.

Theorem

$\theta_H : H_{*,*}(H) \to Hom_{\mathbb{M}_2}(H^{*,*}(H), \mathbb{M}_2)$ is an isomorphism.

The Kunneth formula

For the Kunneth formula, we need to "cellularize" H, the technical details can be found in the "Motivic cell structures". The Kunneth formula can be written as follows:

Theorem

$H^{*,*}(X) \otimes_{\mathbb{M}_2} H^{*,*}(H) \to H^{*,*}(X \wedge H)$ is an isomorphism if X admits a right H-module structure.

Proof of the convergence property

The convergence of the cohomological motivic Adams spectral sequence can be proved by considering its duality tower in the homological range. The details of this proof can be found in the section 6 of "The localization of spectra with respect to homology".

The End