Homotopy theory for digraphs

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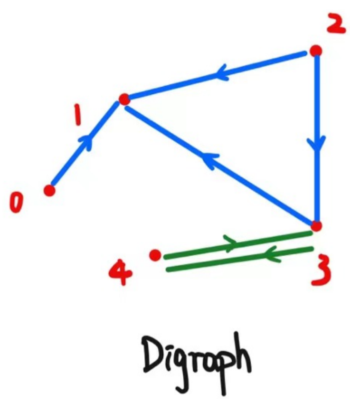
January 5, 2024

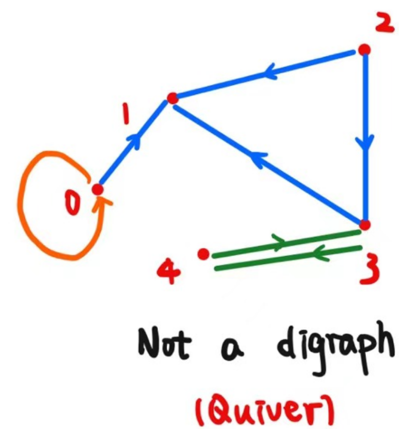
Outline

Basic definitions

Digraph

A digraph (which is also called directed graph) $G = (V, E)$ is a couple of a set V,whose elements are called vertices, and a subset $E \subset \{V \times V \backslash diag\}$ of ordered pairs of vertices that are called edges or arrows. If $v, w \in V, (v, w) \in E$ is also denoted by $v \to w$.





Digraph map

A morphism from $G = (V_G, E_G)$ to $H = (V_H, E_H)$ is a map $f : V_G \to V_H$ such that for any edge $v \to w$ on G we have $f(v) \overset{\rightarrow}{=} f(w)$ on H. (That is either $f(v) \to f(w)$ or $f(v) = f(w)$.) We will refer to such morphisms also as digraph maps and denote them by $f : G \to H$.

- The set of all digraphs with digraph maps form a category of digraphs that will be denoted by $\mathcal{D}$.

Elementary p-path

Let V be a finite set, for any $p \geqslant 0$, an elementary p-path is any ordered sequence $i_0, \cdots, i_p$ of $p+1$ vertices of V denoted by $i_0 \cdots i_p$ or $e_{i_0 \cdots i_p}$.

- Fix a commutative ring $\mathbb{K}$ with unity and denote by $\Lambda_p = \Lambda_p(v) = \Lambda(V, \mathbb{K})$ the free $\mathbb{K}$-module which consists of all formal $\mathbb{K}$-linear combinations of all elementary p-paths.
- Hence,each p-path has a form

$$v = \sum_{i_0, \cdots, i_p} v^{i_0, \cdots, i_p} e_{i_0 \cdots i_p},$$

where $v^{i_0, \cdots, i_p} \in \mathbb{K}$.

Boundary operator

For any $p \geqslant 0$, the boundary operator $\partial : \Lambda_{p+1} \to \Lambda_p$ is defined by

$$\partial v = \sum_{i_0,\cdots,i_p} \Big(\sum_k \sum_{q=0}^{p+1} (-1)^q v^{i_0\cdots i_{q-1} k i_q \cdots i_p}\Big) e_{i_0\cdots i_p},$$

where $v = \sum_{i_0,\cdots,i_{p+1}} v^{i_0,\cdots,i_{p+1}} e_{i_0\cdots i_{p+1}}$.

- $\partial e_{j_0\cdots j_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{j_0\cdots \hat{j}_q \cdots j_{p+1}}$, $\partial^2 v = 0$ for any $v \in \Lambda_p$.
- Set $\Lambda_{-1} = \{0\}$ and $\partial v = 0$ for all $v \in \Lambda_0$ in case we need the operator $\partial : \Lambda_0 \to \Lambda_{-1}$.

Hence, the family of $\mathbb{K}$-modules $\{\Lambda_p\}_{p\geqslant -1}$ with the boundary operator ∂ determine a chain complex denoted by $\Lambda_*(V) = \Lambda_*(V, \mathbb{K})$.

Regular path

An elementary p-path $e_{i_0 \cdots i_p}$ on a set V is called regular if $i_k \neq i_{k+1}$ for all $k = 0, \cdots, p-1$, and irregular otherwise.

- Let I_P be the submodule of Λ_P that is $\mathbb{K}$-spanned by irregular $e_{i_0 \cdots i_p}$, and $\partial I_p \subset I_{p-1}$.
- Consider the quotient $\mathcal{R}_p := \Lambda_p / I_p$, then the induced boundary operator $\partial : \mathcal{R}_p \to \mathcal{R}_{p-1}, p \geqslant 0$ is well-defined. Denote by $R_*(V)$ the obtained chain complex.

Allowed and ∂-invariant path

Let $G = (V, E)$ be a digraph. An elementary p-path $i_0 \cdots i_p$ on V is called allowed if $i_k \to i_{k+1}$ for any $k = 0, \cdots, p-1$, and non-allowed otherwise.

- Note that the modules $\mathcal{A}_p$ are in general not invariant for ∂. So we consider the submodules $\Omega_p := \{v \in \mathcal{A}_p, \partial v \in \mathcal{A}_{p-1}\}$, which are ∂-invariant.
- Hence, we obtain a chain complex $\Omega_* = \Omega_*(G, \mathbb{K})$:

$$\cdots \xrightarrow{\partial} \Omega_p \xrightarrow{\partial} \Omega_{p-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_1 \xrightarrow{\partial} \Omega_0 \xrightarrow{\partial} 0$$

Homologies of a digraph

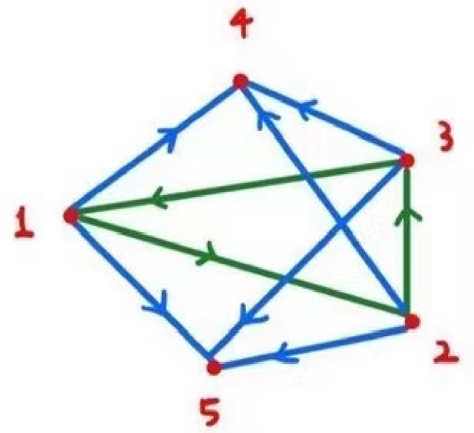
- Define for any $p \geqslant 0$ the homologies of the digraph G with coefficients from $\mathbb{K}$ by

$$H_P(G, \mathbb{K}) = H_p(G) := H_p(\Omega_*(G)) = \ker \partial|_{\Omega_p} / Im\partial|_{\Omega_{p+1}}.$$

$$\cdots \xrightarrow{\partial} \Omega_p \xrightarrow{\partial} \Omega_{p-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_1 \xrightarrow{\partial} \Omega_0 \xrightarrow{\partial} 0$$

Examples

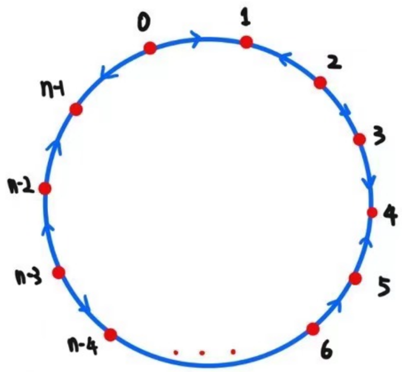
Example 1: Planar digraph with a nontrivial homology group H_2.



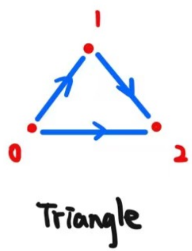
A direct computation:

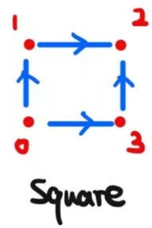
$$H_1(G, \mathbb{K}) = \{0\}, \; H_2(G, \mathbb{K}) \cong \mathbb{K}.$$

Example 2: Cycle digraph $S_n (n \geqslant 3)$.



$H_1(G, \mathbb{K}) \cong \mathbb{K}$ if S_n contains neither triangle nor square below.





Homotopy theory for digraphs

Line digraph

A line digraph is a digraph whose vertices set is $\{0, 1, \cdots, n\}$ and the set of edges contains exactly one of the edges $i \to (i+1)$, $(i+1) \to i$ for any $i = 0, 1, \cdots n-1$, and no other edges.
Denote by $\mathcal{I}_n$ the set of all line digraphs and $\mathcal{I}$ the union of all $\mathcal{I}_n$.

Cartesian product

For two digraphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, define the Cartesian product $G \boxdot H$ as a digraph with the set $V_G \times V_H$ and with the set of edges as follows: for $x, x' \in V_G$ and $y, y' \in V_H$, we have $(x, y) \to (x', y')$ in $G \boxdot H$ iff either $x = x'$ and $y \to y'$, or $x \to x'$ and $y = y'$.

Homotopy theory for digraphs

Homotopy

Let G, H be two digraphs. Two digraph maps $f, g : G \to H$ are called homotopic if there exists a line digraph $I_n \in \mathcal{I}_n$ with $n > 1$ and a digraph map $F : G \boxdot I_n \to H$ such that $F|_{G \boxdot \{0\}} = f$ and $F|_{G \boxdot \{n\}} = g$. The map F is called a homotopy between f and g.

Homotopy equivalent

Two digraphs G and H are called homotopy equivalent if there exists digraph maps $f : G \to H, g : H \to G$ such that $f \circ g \simeq id_H$, $g \circ f \simeq id_G$. The maps f and g are called homotopy inverses of each other.

Homotopy preserves homologies

Theorem

Let G and H be two digraph maps.

(i)Let $f, g : G \to H$ be two homotopic digraph maps, then these maps induce the identical homomorphisms of homology groups of G and H, that is: $f_* : H_p(G) \to H_p(H)$ and $g_* : H_p(G) \to H_p(H)$ are identical.

(ii) If the digraphs G and H are homotopy equivalent, then they have isomorphic homology groups.

Retraction

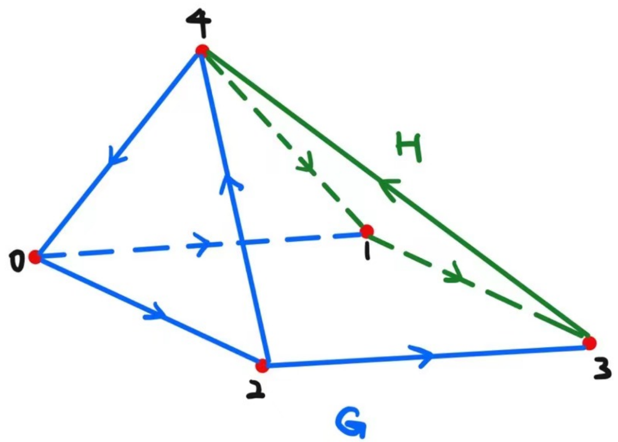
Let G be a digraph and H be its sub-digraph.
(i) A retraction of G onto H is a digraph map $r : G \to H$ such that $r|_H = id_H$.
(ii) A retraction $r : G \to H$ is called a deformation retraction if $i \circ r \simeq id_G$, where $i : H \to G$ is the natural inclusion map.

Corollary

Let $r : G \to H$ be a retraction of a digraph G onto a sub-digraph H and $x \overset{\rightarrow}{=} r(x)$ for all $x \in V_G$ or $r(x) \overset{\rightarrow}{=} x$ for all $x \in V_G$. Then r is a deformation retraction, the digraphs G and H are homotopy equivalent, and i, r are their homotopy inverses.

Examples

Consider the following digraph G and its sub-digraph H.



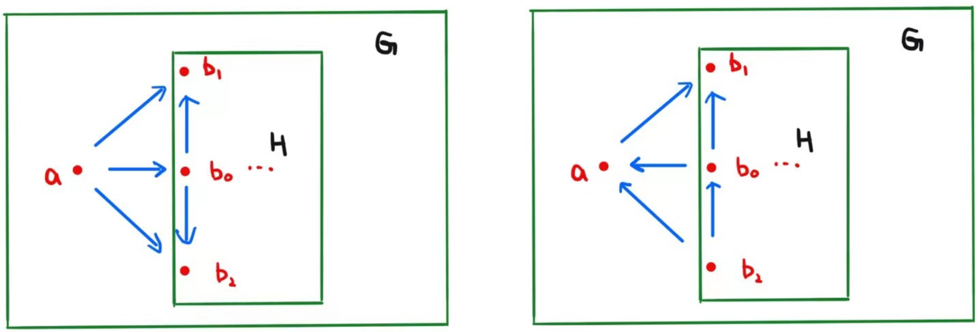
Define a retraction $r : G \to H$ by $r(0) = 1, r(2) = 3, r|_H = id|_H$.
By corollary, r is a deformation retraction, whence, $G \simeq H$. And thus $H_1(G, \mathbb{K}) \cong H_1(H, \mathbb{K}) \cong \mathbb{K}$ and $H_p(H, \mathbb{K}) = \{0\}$ for $p \geqslant 2$.

Lemma

Let a be a vertex in a digraph G and $b_0, b_1, b_2, \cdots, b_n$ be all the neighboring vertices of a in G. Assume that the following condition is satisfied:

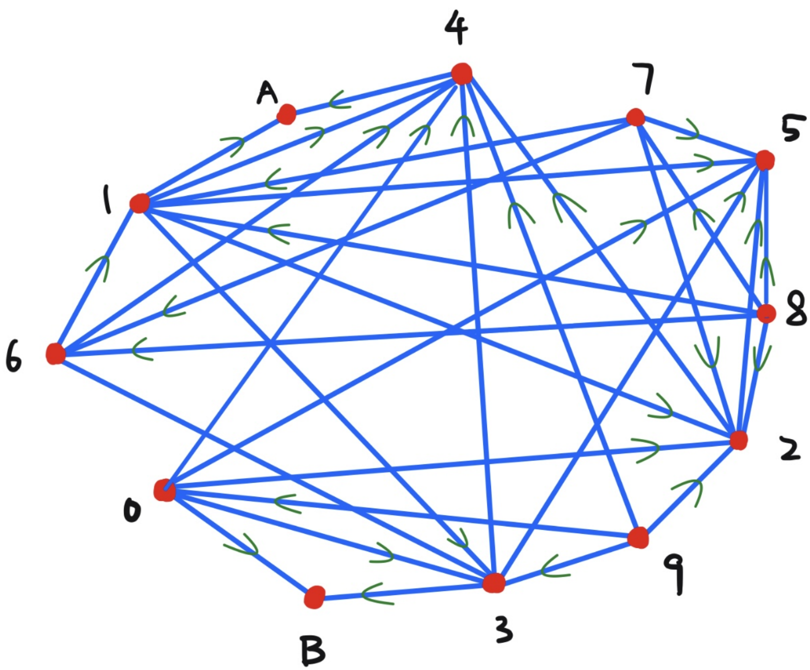
$$\forall i = 1, \cdots, n: \ a \to b_i \Rightarrow b_0 \to b_i,$$

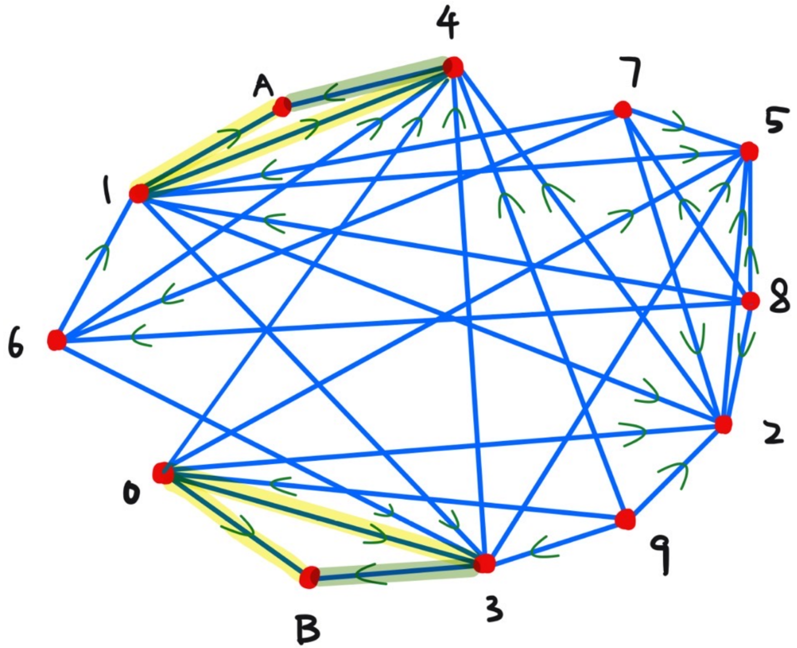
$$\forall j = 1, \cdots, n: \ b_j \to a \Rightarrow b_j \to b_0.$$



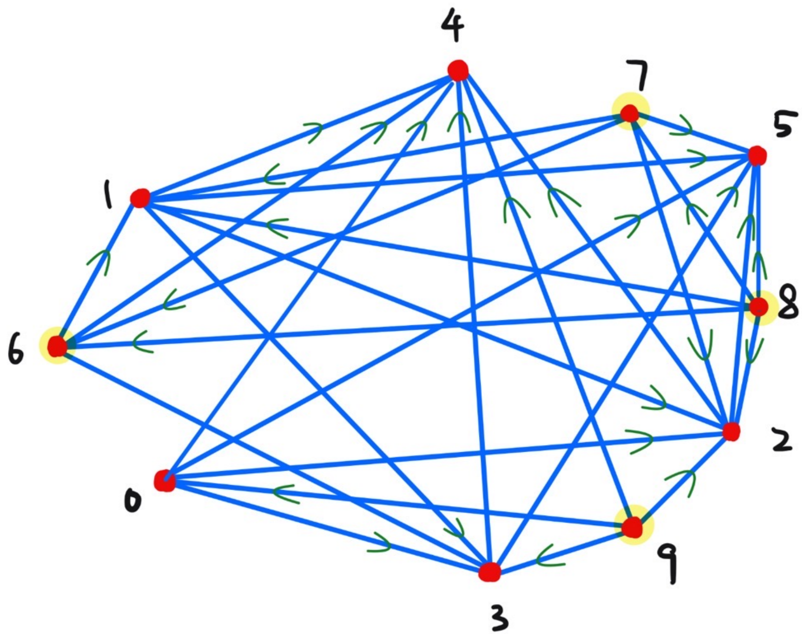
The map $r : G \to H$ given by $r(a) = b_0$ and $r|_H = id_H$ is a deformation retraction, whence $G \simeq H$.

Now consider homologies of a complicated digraph G.

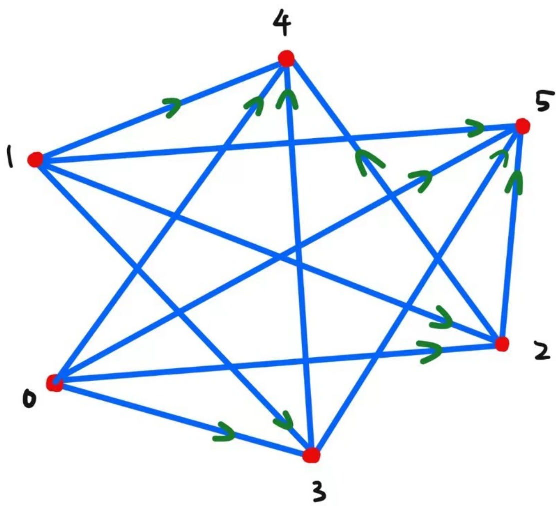




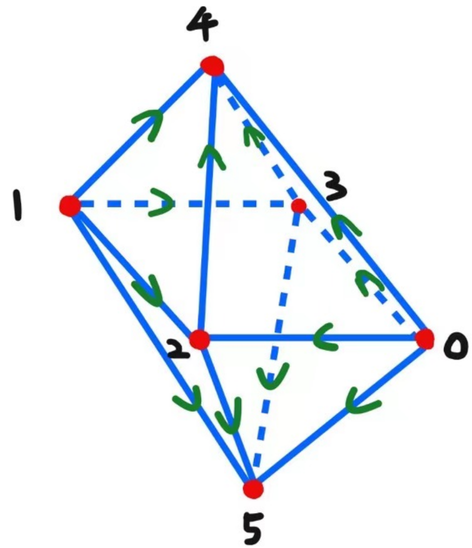
By the Lemma above, we can remove A, B and all their adjacent edges without changing the homologies of G, and thus simplify the computation.



Similarly, we can remove $6, 7, 8, 9$ and all their adjacent edges.



H_1



H_2

$$H_p(G,\mathbb{K}) = H_p(H_1,\mathbb{K}) \begin{cases} = \{0\}, p = 1, p > 2. \\ \cong \mathbb{K}, p = 2. \end{cases} \tag{1}$$

Based digraphs and based digraph maps

A based digraph G^* is a digraph G with a fixed base vertex $* \in V_G$. A based digraph map $f : G^* \to H^*$ is a digraph map $f : G \to H$ such that $f(*) = *$.

- Remark: A homotopy between two based digraph maps $f, g : G^* \to H^*$ is defined as digraph maps with additional requirement that $F|_{\{*\}\boxdot I_n} = *$.

Construction of π_0

Let G^* be a based digraph, and $V_2^* = \{0,1\}$ be the based digraph consisting of two vertices, no edges and with the base vertex $0 = *$. $Hom(V_2^*, G^*)$ is defined to be the set of based digraph maps from V_2^* to G^*.

- Two digraph maps $\phi, \psi \in Hom(V_2^*, G^*)$ are equivalent if there exists $I_n \in \mathcal{I}$ and a digraph map $f : I_n \to G$ such that $f(0) = \phi(1)$ and $f(n) = \psi(1)$.
- And we denote by $[\phi]$ the equivalence class of the element ϕ, and by $\pi_0(G^*)$ the set of classes of equivalence with the base point $*$ given by a class of equivalence of the trivial map $V_2 \to * \in G^*$.

Proposition

Any based digraph map $f : G^* \to H^*$ induces a map $\pi_0(f) : \pi_0(G^*) \to \pi_0(H^*)$ of based sets. In particular, the homotopic maps induce the same map of based sets.

Proof of sketch:

- If $x \sim y$ in $\pi_0(G^*)$, then $\pi_0(f)(x) \sim \pi_0(f)(y)$ in $\pi_0(H^*)$.
- If $f \simeq g : G^* \to H^*$, then $\pi_0(f) = \pi_0(g)$.

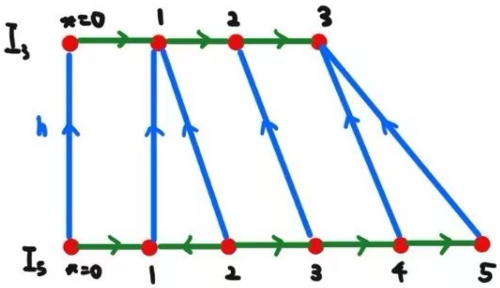
Construction of π_1

Path-map

A path-map in a digraph G is any digraph map $\phi : I_n \to G$, where $I_n \in \mathcal{I}_n$. If $\phi : I_n^* \to G^*$ satisfies $\phi(0) = *$, it is then called a based path-map.

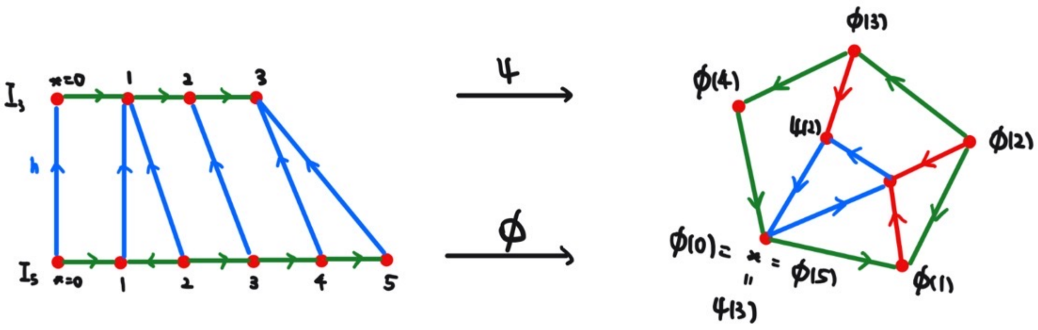
Shrinking map

A digraph map $h : I_n \to I_m$ is called shrinking map if $h(0) = 0$, $h(n) = m$, and $h(i) \leqslant h(j)$ whenever $i \leqslant j$.



C-homotopy

Consider two based path-maps $\phi : I_n^* \to G^*$ and $\psi : I_m^* \to G^*$. An one-step C-homotopy from ϕ to ψ is given by a shrinking map $h : I_n \to I_m$ such that $\phi(i) \overrightarrow{=} \psi(h(i))$ for all $i \in I_n$.



Construction of π_1

C-homotopic

Two based path-maps ϕ, ψ in a digraph G are called C-homotopic if there exists a finite sequence $\{\phi_k\}_{k=0}^{m}$ of based path-maps such that $\phi_0 = \phi$, $\phi_m = \psi$, and for any $k = 0, \cdots, m-1$ there exists an one-step C-homotopy between ϕ_k and ϕ_{k+1}.

π_1 of a digraph

Let $\pi_1(G^*)$ be a set of equivalence classes under C-homotopy of based loops of a diagraph G^*. The C-homotopy class of a based loop ϕ will be denoted by $[\phi]$.

Group Structure in π_1

- For two path-map $\phi : I_n \to G$ and $\psi : I_m \to G$ with $\phi(n) = \psi(0)$ define the concatenation path-map $\phi \vee \psi : I_{m+n} \to G$ by

$$(\phi \vee \psi)(i) = \begin{cases} \phi(i), & 0 \leqslant i \leqslant n. \\ \psi(i-n), & n \leqslant i \leqslant n+m. \end{cases} \tag{2}$$

  For any two loops $\phi : I_n^* \to G^*$ and $\psi : I_m^* \to G^*$ define the product of $[\phi]$ and $[\psi]$ by $[\phi] \cdot [\psi] = [\phi \vee \psi]$.

- For a path-map $\phi : I_n \to G$ define the inverse path-map $\hat{\phi} : \hat{I}_n \to G$ by $\hat{\phi}(i) = \phi(n-i)$.
- For any loop $\phi : I_n^* \to G^*$ we have $\phi \vee \hat{\phi} \overset{C}{\simeq} e$ where $e : I_0^* \to G^*$ is the trivial loop.

Theorem

Let G, H be digraphs.
(i) The set $\pi_1(G^*)$ with the product and neutral element $[e]$ is a group.
(ii) A based digraph map $f : G^* \to H^*$ induces a group homomorphism $\pi_1(f) : \pi_1(G^*) \to \pi_1(H^*), (\pi_1(f))[\phi] \mapsto [f \circ \phi]$, which depends only on homotopy class of f.
(iii) Let $\gamma : I_k^* \to G^*$ be a based path-map with $\gamma(k) = v$. Then γ induces an isomorphism of fundamental groups $\gamma_\# : \pi_1(G^*) \to \pi_1(G^v)$, which depends only on C-homotopy class of the path-map γ.

Relation between H_1 and π_1

Theorem

Let G, H be two connected digraphs. If $G \simeq H$ then the fundamental groups $\pi_1(G^*)$ and $\pi_1(H^*)$ are isomorphic.

Theorem

For any based connected digraph G^* we have an isomorphism

$$\pi_1(G^*)/[\pi_1(G^*), \pi_1(G^*)] \cong H_1(G, \mathbb{Z}),$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is a commutator subgroup.

Some applications of digraphs

- Sperner coloring.
- Topological data analysis.

Wu, Shuang, et al. "The Metabolomic Physics of Complex Diseases." Proceedings of the National Academy of Sciences - PNAS, https://doi.org/10.1073/pnas.2308496120.

References

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2.Jimenez, R Vershinin, V Muranov, Y. (2023). On Cubical Sets of Quivers and Digraphs. arXiv.2310.00634.