Elliptic homology theory and *tmf*

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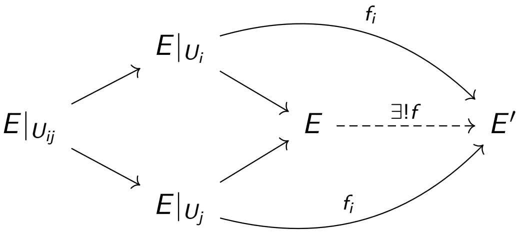
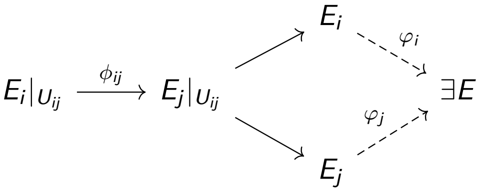
(1) (Gluing morphisms) For $E, E' \in F(X)$, and $f_i : E|_{U_i} \to E'|_{U_i}$ such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$,



there is a unique isomorphism $f : E \to E' \in F(X)$ such that $f_i = f|_{U_i}$.

(2) (Gluing objects) For $E_i \in F(U_i)$ and isomorphisms $\phi_{ij} : E_i|_{U_{ij}} \to E_j|_{U_{ij}}$ with $\phi_{jk} \circ \phi_{ij} = \phi_{ik}|_{U_{ijk}}$,



there is an object $E \in F(X)$ together with isomorphisms $\varphi_i : E|_{U_i} \to E_i$ such that $\phi_{ij} \circ \varphi_i = \varphi_j$.

Definition

A morphism $\mathcal{F} \to \mathcal{G}$ of stack on $Sch_{/S}$ is called representable if for every morphism $X \to \mathcal{G}$ with X a scheme, the pullback $F \times_{\mathcal{G}} X$ is a scheme.

A stack $\mathcal{M}$ is algebraic for the fpqc topology on Sch if there is a cover Spec $A \to \mathcal{M}$ for some A.

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A stack $\mathcal{M}$ is algebraic for the fpqc topology on Sch if there is a cover Spec $A \to \mathcal{M}$ for some A.

Let P be a property of morphisms of schemes that is closed under pullback,

$$\begin{array}{ccc} \mathcal{F} \times_{\mathcal{G}} X & \longrightarrow & \mathcal{F} \\ \downarrow P & \ulcorner & \downarrow P \\ X & \longrightarrow & \mathcal{G} \end{array}$$

then we say a representable morphism of stacks $\mathcal{F} \to \mathcal{G}$ is P if $\mathcal{F} \times_{\mathcal{G}} X$ satisfies P for every $X \to \mathcal{G}$ with X a scheme.

For any scheme X, we define

- $\mathcal{M}_{ell}(X)$: groupoid of elliptic curves over X with isomorphisms of elliptic curves over X,
- $\mathcal{M}_{fg}(X)$: groupoid of 1-dim formal groups over X with isomorphisms of formal groups over X.

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Proposition

(1) $\mathcal{M}_{fg}$ is algebraic with a cover Spec $L \to \mathcal{M}_{fg}$.
(2) $\mathcal{M}_{ell}$ is algebraic with a cover Spec $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}] \to \mathcal{M}_{ell}$.

Landweber Exact Functor Theorem

Let F be a formal group law over a graded ring E_*, and consider the functor

$$h_F : X \mapsto MU_*(X) \otimes_{MU_*} E_*$$

from hSp to graded abelian groups. When is h_F a homology theory?

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Theorem (Landweber)

h_F is a homology theory if for every p, the sequence $(p, v_1, \dots)$ is regular on E_*, i.e. that $v_i : E_*/(p, v_1, \dots v_{i-1}) \to E_*/(p, v_1, \dots v_{i-1})$ is injective.

In the language of stack

Proposition

Let M be a graded module over the Lazard ring. M is Landweber exact if and only if M is flat over $\mathcal{M}_{fg}$.

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The flatness means that for every η,

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} L \\ \downarrow & \ulcorner & \downarrow \\ \operatorname{Spec} R' & \xrightarrow{\ \eta\ } & \mathcal{M}_{fg} \end{array}$$

$M \otimes_L B$ is flat over R'.

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$M \otimes_L B$ is flat over R'.

Theorem

Let M be a module over L, then M is flat over $\mathcal{M}_{fg}$ if and only if for every prime p, the sequence $(p, v_1, \dots)$ on M is regular.

Proposition

Let R be a (ungraded) commutative ring and $q : \mathrm{Spec}\, R \to \mathcal{M}_{fg}$ be a flat map. Then there existsan even periodic ring spectrum E_R such that $(E_R)_*(X) = MU_*(X) \otimes_L R[u^{\pm 1}]$. In particular, $(E_R)_0(X) = MU_{even}(X) \otimes_L R$.

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For example, let R be the Lazard ring L with the canonical morphism Spec $L \to \mathcal{M}_{fg}$, then the above construction applies to produce a spectrum E_L s.t.

$$(E_L)_*(X) = MU_*(X)[u^{\pm 1}].$$

E_L is called the periodic complex bordism spectra and is denoted by MP.

Elliptic cohomology

Proposition

The morphism $\mathcal{F} : \mathcal{M}_{ell} \to \mathcal{M}_{fg}$ that maps an elliptic curve C to the associated formal group F_C is flat.

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Thus, given a flat map $C : \text{Spec } R \to \mathcal{M}_{ell}$, we can define the elliptic homology theory associated with C by

$$Ell^C_*(X) := MP_*(X) \otimes_L R.$$

This construction gives a presheaf

$$Ell : (Aff/\mathcal{M}_{ell})_{flat} \to hSp.$$

$$(Aff/\mathcal{M}_{ell})_{flat} \xrightarrow{Ell} hSp \qquad \begin{array}{ccc} & & Sp \\ & \overset{\mathcal{O}^{top}}{\nearrow} & \downarrow \\ (\overline{\mathcal{M}_{ell}})_{\text{ét}} & \xrightarrow{Ell} & hSp \end{array}$$

Here Sp can be the category of S-module/ symmetric spectra/ orthogonal specta.

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Theorem

There is a presheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on the site $(\overline{\mathcal{M}}_{ell})_{\acute{e}t}$. Given an affine étale open $Spec\ R \xrightarrow{C} \overline{\mathcal{M}}_{ell}$ classifying a generalized elliptic curve C/R, $E = \mathcal{O}^{top}(Spec\ R)$ is a weakly even periodic ring spectrum satisfying:
(1) $\pi_0(E) \cong R$,
(2) $\mathbb{G}_E \cong \hat{C}$.

Sheaves of spectra

Let $\mathcal{C}$ be a site. A sheaf on $\mathcal{C}$ with values in the category Sp is a presheaf F s.t. for all covers $\{U_i \to U\}_{i \in I}$, the map

$$F(U) \to \text{holim}(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \dots)$$

is a weak equivalence.

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Let $PreSp_{\mathcal{C}}$ denote the category of presheaves of symmetric spectra, then $PreSp_{\mathcal{C}}$ has a Jardine model category structure, where

- the cofibrations are the sectionwise cofibrations,
- the weak equivalences are the stalkwise weak equivalence,
- The fibrant objects satisfy descent.

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$$(\mathcal{M}_{ell})_{\text{ét}} = \begin{cases} \text{obj} = \{\mathcal{N} \xrightarrow{\text{ét}} \mathcal{M}_{ell}\} \\ \text{mor} = \left\{ \begin{array}{c} \mathcal{N} \xrightarrow{a} \mathcal{M}_{ell} \\ c \downarrow \;\; \phi \Downarrow \\ \mathcal{N}' \xrightarrow{b} \mathcal{M}_{ell} \end{array} \right\} \Big/ \left\{ \begin{array}{c} \mathcal{N} \\ d \downarrow \overset{\psi}{\Leftarrow} \downarrow c \\ \mathcal{N}' \end{array} \right\} \\ \text{cov} = \left\{ \left\{ \begin{array}{c} \mathcal{N}_i \to \mathcal{M}_{ell} \\ c_i \downarrow \;\; \phi_i \Downarrow \\ \mathcal{N} \to \mathcal{M}_{ell} \end{array} \right\} \text{ s.t. } \begin{array}{c} \exists \tilde{f} \nearrow \mathcal{N}_i \\ \qquad\quad \downarrow \\ \operatorname{Spec} k \xrightarrow[\forall f]{} \mathcal{N} \end{array} \right\} \end{cases}$$

$(\overline{\mathcal{M}}_{ell})_{\text{ét}}$

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Let $i : (Aff/\overline{\mathcal{M}}_{ell})_{\text{ét}} \to (\overline{\mathcal{M}}_{ell})_{\text{ét}}$ be the full subcategory consisting of only the affine étale opens.

Proposition

$i : (Aff/\overline{\mathcal{M}}_{ell})_{ét} \to (\overline{\mathcal{M}}_{ell})_{ét}$ induces an adjoint pair which is a Quillen equivalence.

$$i^* : PreSp_{(\overline{\mathcal{M}}_{ell})_{ét}} \rightleftarrows PreSp_{(Aff/\overline{\mathcal{M}}_{ell})_{ét}} : i_*.$$

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Thus, it suffices to construct a sheaf on $(Aff/\overline{\mathcal{M}}_{ell})_{ét}$.

$$\mathcal{F} \in PreSh_{(Aff/\overline{\mathcal{M}}_{ell})_{ét}} \xrightarrow{fibrant} \mathcal{F}' \xrightarrow{i_*} \mathcal{G} := i_*\mathcal{F}'.$$

If $\mathcal{F}$ is a sheaf, then there is a zigzag of sectionwise weak equivalence $i^*\mathcal{G} \to \mathcal{F}' \leftarrow \mathcal{F}$.

Descent lemma

Lemma

Suppose that $\mathcal{F} \in PreSh_{(Aff/\overline{\mathcal{M}_{ell}})_{ét}}$ and suppose that there is a graded quasi-coherent sheaf $\mathcal{A}_*$ on $\overline{\mathcal{M}_{ell}}$ and natural isomorphisms

$$f_U : \mathcal{A}_*(U) \to \pi_*\mathcal{F}(U)$$

for all affine étale opens $U \to \overline{\mathcal{M}_{ell}}$. Then $\mathcal{F}$ satisfies homotopy descent.

It suffices to construct $\mathcal{O}^{top}(\text{Spec } R)$ for all étale Spec $R \to \overline{\mathcal{M}_{ell}}$.

Decomposition of $\overline{\mathcal{M}}_{ell}$

Lemma

Suppose that C is a generalized elliptic curve over a ring R, and that E is an elliptic spectrum associated with C. Then E is $E(2)$-local.

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$$\begin{array}{ccc} \mathcal{O}^{top} & \longrightarrow & \prod_p (\iota_p)_* \mathcal{O}^{top}_p \\ \downarrow & & \downarrow \\ (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}} & \longrightarrow & (\prod_p (\iota_p)_* \mathcal{O}^{top}_p)_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} \overline{\mathcal{M}}_{ell} & \xleftarrow{\iota_p} & (\overline{\mathcal{M}}_{ell})_p \\ \uparrow{\scriptstyle \iota_{\mathbb{Q}}} & & \\ (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} & & \end{array}$$

$$(\overline{\mathcal{M}}_{ell})_p = p - \text{completion of } \overline{\mathcal{M}}_{ell},$$
$$(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} = \overline{\mathcal{M}}_{ell} \otimes \mathbb{Q}.$$

$$\begin{array}{ccc} \mathcal{O}^{top}_p & \longrightarrow & (\iota_{ss})_*\mathcal{O}^{top}_{K(2)} \\ \downarrow & & \downarrow \\ (\iota_{ord})_*\mathcal{O}^{top}_{K(1)} & \xrightarrow{\alpha} & ((\iota_{ss})_*\mathcal{O}^{top}_{K(2)})_{K(1)} \end{array} \qquad \begin{array}{ccc} (\overline{\mathcal{M}_{ell}})_p & \xleftarrow[\iota_{ss}]{} & \mathcal{M}^{ss}_{ell} \\ {\scriptstyle \iota_{ord}}\uparrow & & \\ \mathcal{M}^{ord}_{ell} & & \end{array}$$

$\mathcal{M}^{ord}_{ell}$ =substack of generalized elliptic curves over p-complete ring with ordinary reduction,

$\mathcal{M}^{ss}_{ell}$ =p-completion of substack of supersingular elliptic curves.

E_∞ Realization problem

Let

- E_*: homotopy commutative ring spectrum satisfying the Adams condition,
- $\mathcal{O}$: the category of operad,
- $\mathcal{F} \in \mathcal{O}$: E_∞-operad,
- A: $E_*\mathcal{F}$ algebra in E_*E-comodules,

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$$E_*X \cong A,$$

morphisms are morphisms of $\mathcal{F}$-spectra which are E_*-isomorphisms.

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  morphisms are morphisms of $\mathcal{F}$-spectra which are E_*-isomorphisms.

The moduli space $\mathcal{TM}(A)$ of all realization of A in $Alg_{\mathcal{F}}$ is $|N\mathcal{E}(A)|$.

Simplicial Operad

We want a postnikov system of E_∞-ring spectra to build resolutions of $\mathcal{TM}(A)$, to construct $X \in \mathcal{TM}(A)$ by inductively.

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Theorem

Let C be an E_∞-operad. Then there exists an argumented simplicial operad $T \to C$ s.t.

- *Each $T_n(k)$ is Σ_k-free.*
- *$|T| \to C$ is a weak equivalence.*

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- *Each $T_n(k)$ is Σ_k-free.*
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Let $sAlg_T$ be the category of simplicial T-algebras in spectra.

$$X \in sAlg_T \implies |X| \in Alg_{|T|}.$$

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Proposition (The spiral exact sequence)

There is a long exact sequence :

$$\to \pi_{p-1,q+1}(E \wedge X) \to \pi_{p,q}(E \wedge X) \to \pi_p E_q X \to \pi_{p-2,q+1}(E \wedge X) \to$$

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Theorem

Let $\mathcal{TM}_\infty(A)$ be the moduli space of simplicial T-algebra X s.t. $\pi_* E_* X = \pi_0 E_* X = A$, then the geometric realization functor gives a weak equivalence $\mathcal{TM}_\infty(A) \simeq \mathcal{TM}(A)$.

Postnikov system for simplicial algebras in spectra

By the spiral exact sequence:

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If $X \in \mathcal{TM}_\infty(A)$, then $\pi_{p,*}(E \wedge X) \cong \Omega^p(A)$ for all $p \geq 0$.

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Definition

Let $X \in sAlg_T$. A Postnikov tower for X is a tower of simplicial T-algebras under X

$$X \to \cdots \to P_n X \to P_{n-1} X \to \cdots \to P_0 X,$$

s.t. for every $f : X \to P_n X$,

$$f_* : \pi_{i,*}(E \wedge X) \xrightarrow{\cong} \pi_{i,*}(E \wedge P_n X), \quad i \leq n,$$

and s.t. $\pi_{i,*}(E \wedge P_n X) = 0$ for $i > n$.

Decomposition of $\mathcal{TM}_\infty(A)$

Definition

Let $X \in sAlg_T$. We say that X is a potential n-stage for A if

$$\pi_i E_* X \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq n+1 \end{cases}$$

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and $\pi_{i,*}(E \wedge X) = 0$ for $i > n$.

Let $\mathcal{TM}_n(A)$ be the moduli space of potential n-stages for A, then the Postnikov section induces

$$\mathcal{TM}_\infty(A) \to \cdots \to \mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A) \to \ldots \mathcal{TM}_0(A).$$

Let $B_A(M, n)$ be an E_∞ spectrum with $\pi_{0,*} = A$ and $\pi_{n,*} = M$ for an A-module M.

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Proposition

There is a natural map $E_* B_A(M,n) \to K_A(M,n)$ which induces an isomorphism

$$\hom_{sAlg_T/B_A}(X, B_A(M,n)) \to \hom_{sAlg_{E_*T/E_*E}/A}(E_*(X), K_A(M,n)).$$

Here $K_A(M,n) = K(M,n) \rtimes A \in sAlg_{E_*T/E_*E}$.

Let

$$\mathcal{H}^n(A; M) = \hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(M, n)),$$

$$\hat{\mathcal{H}}^n(A; M) = \mathcal{H}^n(A; M) \times_{Aut(A,M)} EAut(A, M).$$

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Theorem

The following is a homotopy pullback square:

$$\begin{array}{ccc} \mathcal{TM}_n(A) & \longrightarrow & BAut(A, \Omega^n A) \\ \downarrow & & \downarrow \\ \mathcal{TM}_{n-1}(A) & \longrightarrow & \hat{\mathcal{H}}^{n+2}(A; \Omega^n A) \end{array}$$

Obstruction theory

Let $Y \in \mathcal{TM}_{n-1}(A)$, then

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Corollary

There are obstructions $\theta_n \in \hom_{sAlg_{E_*T/E_*E}/A}(A, K_A(M, n))$ to existance of a commutative S-algebra X with $E_*X \cong A$.

p-adic K theory

Let E be a $K(1)$ local weakly even periodic ring spectrum, and let

$$(K_p^\wedge)_* E := \pi_*((K \wedge E)_p)$$

denote the p-adic K-homology of E.

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Theorem

$(K_p^\wedge)_* E$ is naturally a theta-algebra. That is, it has actions of operators

- φ^k, $k \in \mathbb{Z}_p^\times$,
- φ^p, *lift of the Frobenius,*
- θ *satisfying* $\varphi^p(x) = x^p + p\theta(x)$.

Obstruction theory fot $K(1)$-local E_∞-ring spectra

Let k be a θ-algebra, A be a θ-algebra under k, and let M be a A-module.

$$H^n_{Alg_\theta}(A/k, M) = \pi_0 \mathrm{map}_{sAlg_\theta^k}(A, K_A(M, n)).$$

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Theorem

The obstructions to the existance of a $K(1)$-local E_∞-ring spectrum E such that $(K_p^\wedge)_* E \cong A_*$ of θ-algebras, lie in

$$H^s_{Alg_\theta}(A_*/(K_p)_*, \Omega^{s-2} A_*), \quad s \geq 3.$$

The obstructions to uniqueness lie in

$$H^s_{Alg_\theta}(A_*/(K_p)_*, \Omega^{s-1} A_*), \quad s \geq 2.$$

Let E_1, E_2 be $K(1)$-local E_∞-ring spectra such that $(K_p^\wedge)_* E_i$ is p-complete, and $f : (K_p^\wedge)_* E_1 \to (K_p^\wedge)_* E_2$ be a map of graded θ-algebras.

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- The obstructions to the existance of a map $f : E_1 \to E_2$ of E_∞-ring spectra which induces f_* lie in

$$H^s_{Alg_\theta}((K_p^\wedge)_* E_1/(K_p)_*, \Omega^{s-1}(K_p^\wedge)_* E_i), \quad s \geq 2.$$

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- Given such a map f above, there is a spectral sequence:

$$H^s_{Alg_\theta}((K_p^\wedge)_* E_1/(K_p)_*, (K_p^\wedge)_* E_2[t]) \Rightarrow \pi_{-t-s}(E_\infty(E_1, E_2), f).$$

The Igusa tower

$\mathcal{M}^{ord}_{ell}(p^k)$: moduli stack whose R-points is the groupoid of pairs (C, η) where

- C/R: ordinary generalized elliptic curves,
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Proposition

- *Every map $\mathcal{M}^{ord}_{ell}(p^{k+1}) \to \mathcal{M}^{ord}_{ell}(p^k)$ is an étale Z/p-torsor ($(Z/p)^\times$-torsor at $k = 0$).*

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- *If $k \geq 1$ ($k \geq 2$ if $p = 2$), $\mathcal{M}^{ord}_{ell}(p^k) = Spf(V_k)$ for some p-complete ring V_k.*
- *$\mathcal{M}^{ord}_{ell}(p^\infty) = Spf(V^\wedge_\infty)$, and $V^\wedge_\infty$ is a θ-algebra.*

Proposition

$$\begin{array}{ccc} \mathcal{M}^{ord}_{ell}(p^\infty) & \longrightarrow & Spf((K_p^\wedge)_0) \\ \downarrow & & \downarrow \\ \mathcal{M}^{ord}_{ell} & \longrightarrow & \mathcal{M}_{fg} \end{array}$$

is a pullback.

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If there is an elliptic spectrum E associated to an ordinary elliptic curve $C : Spf(R) \to \mathcal{M}^{ord}_{ell}$, then

$$(K_p^\wedge)_0 E \cong W,$$

here $Spf(W) = Spf(R) \times_{\mathcal{M}^{ord}_{ell}} \mathcal{M}^{ord}_{ell}(p^\infty)$.

Outline the construction of $\mathcal{O}^{top}_{K(1)}$

- First construct $tmf_{K(1)} = \mathcal{O}^{top}_{K(1)}(\mathcal{M}^{ord}_{ell})$ as follows: If $p > 2$, then $\mathcal{M}^{ord}_{ell}(p) = Spf(V_1)$, we construct

$$tmf(p)^{ord} = \mathcal{O}^{top}_{K(1)}(V_1)$$

along with an action of $(\mathbb{Z}/p)^\times$ through E_∞-ring maps. Set $tmf_{K(1)} = (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}$.

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- Then construct $\mathcal{O}^{top}_{K(1)}$ in the category of $tmf_{K(1)}$-algebras.

$tmf_{K(1)}$

- Since $\mathcal{M}_{ell}^{ord}(p) = Spf(V_1)$ is formally affine, we can pullback over $\mathcal{M}_{ell}^{ord}(p^\infty)$ to get $(K_p^\wedge)_*(tmf(p)^{ord})$, then the vanishing of obstruction groups guarantee the existance of $tmf(p)^{ord}$.

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- $(K^\wedge_p)_0(tmf(p)_{K(1)}) = V^\wedge_\infty$.
- If $p = 2$, then we replace K with KO, and construct $tmf_{K(1)}$ directly s.t. $(KO^\wedge_2)_*(tmf(p)_{K(1)}) = KO_* \otimes V^\wedge_\infty$.

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There is a relative form of obstruction theory for $K(1)$ local E_∞-ring if we work in the category of $K(1)$ local commutative E-algebras: The obstructions live in $H^s_{Alg^\theta_{W_*}}(A_*, M_*)$ where $W_* = (K_p^\wedge)_* E$.

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Lemma

Suppose that W_* and A_* are even periodic, and that A_0 is étale over W_0. Then for all s,

$$H^s_{Alg^\theta_{W_*}}(A_*, M_*) = 0.$$

Let $Spf(R) \xrightarrow{f} \mathcal{M}_{ell}^{ord}$ be an étale formal affine open, consider the pullback

$$\begin{array}{ccc} Spf(W) & \longrightarrow & \mathcal{M}_{ell}^{ord}(p^\infty) \\ \downarrow & & \downarrow \\ Spf(R) & \longrightarrow & \mathcal{M}_{ell}^{ord}. \end{array}$$

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All obstructions vanish.
Given a map $Spf(R_2) \to Spf(R_1)$:

- $[E_1, E_2]_{Alg_{tmf_{K(1)}}} \cong \hom_{Alg^\theta_{(V^\wedge_\infty)_*}}((W_1)_*, (W_2)_*)$.
- The mapping spaces $Alg_{tmf_{K(1)}}(E_1, E_2)$ have contractible components.

We get a presheaf
$\mathcal{O}^{top}_{K(1)} : (Aff/(\mathcal{M}^{ord}_{ell})_{\text{ét}})^{op} \to$ Commutative $tmf_{K(1)}$ algebras.