Outline :

1. Classifying Spaces

2. (B, f)- structure

3. Manifolds with (B, f)- structure

4. Pontrjagin-Thom isomorphism and Thom Spectra

1. Classifying spaces.

$$O(A) := \{ \text{all isometries } f: A \to A \}$$

Stiefel Manifold — $V_{A,B} = O(A)/O(B)$

where $B \subseteq A$ are vector spaces. If we denote $B^{\perp}$ in A

by C, then $V_{A,B}$ can be identified with the set of all C-frames

in A, i.e. all isometries from C into A.

Grassmanian Manifold — $G_{A,B} = O(A)/O(B) \times O(C)$

$O(B) \times O(C)$ is identified with a subgroup of $O(A)$ via

$$(f,g) \longmapsto f \oplus g \in O(A)$$

$G_{A,B}$ can be identified with the set of all c-dim subspace of A, $c = \dim C$.

There is a canonical map, which is an $O(C)$-bundle.

$$p_{A,B} : V_{A,B} \to G_{A,B}$$

And associated C-vector bundle

$$P_{A,B} : E_{A,B} = V_{A,B} \times_{O(C)} C \to G_{A,B},$$

where $E_{A,B}$ can be identified with $\{ (S,x) \mid S \in G_{A,B},\ x \in S \}$

— Gauss map

$e: M^n \to A$ an embedding,

$\nu_e :=$ the normal bundle of $e(M^n) \subseteq A \times A$.

Choose arbitrary $B \subseteq A$, with $\dim B = n$, then there is a bundle map (called the Gauss map)

$$\gamma_e : \nu_e \to P_{A,B}$$

Suppose we have $B \subseteq A \subseteq A_1$, then we have

$$\begin{tikzcd} V_{A,B} \arrow[r, "i'_{A_1,A,B}"] \arrow[d, "p_{A,B}"'] \arrow[dr, phantom, "\circlearrowright"] & V_{A_1, B+A^\perp} \arrow[d, "p_{A_1,B+A^\perp}"] \\ G_{A,B} \arrow[r, "i''_{A_1,A,B}"'] & G_{A_1, B+A^\perp} \end{tikzcd}$$

where $i'_{A_1,A,B}$: a C-frame f in $A \mapsto$ C-frame f in A_1,

and $i''_{A_1,A,B}$: c-dim subspaces of $A \mapsto \cdots$ in A_1.

$I_{A_1,A,B} : p_{A,B} \to p_{A_1,B+A^\perp}$ is an $O(C)$-bundle map.

Taking the direct limit over all $A \subseteq \mathbb{R}^\infty$ containing C,

we have the universal $O(C)$ - bundle

$$p_C : EO(C) \longrightarrow BO(C).$$

$$\begin{tikzcd} V_{A,B} \arrow[r, "j'_{A_1,A,B}"] \arrow[d, "p_{A,B}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{A_1,B} \arrow[d, "p_{A_1,B}"] \\ G_{A,B} \arrow[r, "j''_{A_1,A,B}"'] & G_{A_1,B} \end{tikzcd}$$

$$j'_{A_1,A,B} : \quad C\text{- frame in } A \mapsto (C + A^\perp)\text{- frame in } A_1$$

$$f \qquad \mapsto \qquad f + 1_{A^\perp}$$

$j''_{A_1,A,B}$: $\quad U \subseteq A \quad \mapsto \quad U + A^{\perp} \subseteq A_1$

Thus taking direct limit over all A containg C and $A_1 = A \oplus D$

we have

$$\begin{tikzcd} EO(C) \arrow[r, "j'_{C,C+D}"] \arrow[d, "P_C"'] \arrow[dr, phantom, "\circlearrowright"] & EO(C+D) \arrow[d, "P_{C+D}"] \\ BO(C) \arrow[r, "j''_{C,C+D}"'] & BO(C+D) \end{tikzcd}$$

For A, C orthogonal in $\mathbb{R}^{\infty}$

$$\begin{tikzcd} V_{B,A^{\perp}} \times V_{D,C^{\perp}} \arrow[r, "w'_{A,B,C,D}"] \arrow[d, "P_{B,A^{\perp}} \times P_{D,C^{\perp}}"'] \arrow[dr, phantom, "\circlearrowright"] & V_{B+D,(A+C)^{\perp}} \arrow[d, "P_{B+D,(A+C)^{\perp}}"] \\ G_{B,A^{\perp}} \times G_{D,C^{\perp}} \arrow[r, "w''_{A,B,C,D}"] & G_{B+D,(A+C)^{\perp}} \end{tikzcd}$$

$$\begin{tikzcd}
EO(A) \times EO(C) \arrow[r, "w'_{A,C}"] \arrow[d, "p_A \times p_C"'] \arrow[dr, phantom, "\circlearrowright"] & EO(A+C) \arrow[d, "p_{A+C}"] \\
BO(A) \times BO(C) \arrow[r, "w''_{A,C}"] & BO(A+C)
\end{tikzcd}$$

— Some property

(a) $EO(C)$ is contractible

[$V_{A,C^\perp}$ is $(a-c-1)$-connected $a=\dim A$, $c=\dim C$.

taking direct limit we have all homotopy groups of $EO(C)$ venish, and $EO(C)$ is a CW-complex, thus contractible]

(b) A, C orthogonal, $a=\dim A$, $c=\dim C=1$, then

$$S^a \longrightarrow BO(A) \xrightarrow{\overset{''}{j}_{A,A+C}} BO(A+C)$$

$$\begin{array}{ccccc} [\quad O(A+C)/O(A) & \longrightarrow & EO(A+C)/O(A) & \longrightarrow & EO(A+C)/O(A+C) \\ \| & & \| & & \| \\ S^a & & BO(A) & & BO(A+C) \end{array}$$

$$\begin{tikzcd}
O(A) \arrow[r, "1"] \arrow[d] & O(A) \arrow[d] \\
EO(A) \arrow[r, "j'_{A,A+C}"] \arrow[d] & EO(A+C) \arrow[d] \\
BO(A) \arrow[r, dashed, "j"] \arrow[dr, "j''_{A,A+C}"'] & EO(A+C)/O(A) \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\
& EO(A+C)/O(A+C)
\end{tikzcd}$$

I

2. (B, f) – structure

$\mathfrak{A}$: poset of all finite dim subspaces of $\mathbb{R}^\infty$,

closed under sum

$\mathfrak{C}$: cofinal subset of $\mathfrak{A}$, morphism isometries.

A (B, f) – structure $\mathfrak{B} = (B, f, \lambda)$

(cellular maps)

1. B – functor $\mathfrak{C} \longrightarrow$ based CW complexes

2. λ – $\lambda_{A,C} : B_A \longrightarrow B_C$ $\quad A \subseteq C$ in $\mathfrak{C}$

3. f – natural transformation, $f : B \to BO$

and $f_A : B_A \longrightarrow BO(A)$ is a fibration

$$\begin{tikzcd} B_A \arrow[r, "\lambda_{A,C}"] \arrow[d, "f_A"'] & B_C \arrow[d, "f_C"] \\ BO(A) \arrow[r, "j''_{A,C}"] & BO(C) \end{tikzcd}$$

Furthermore, a multiplicative $\mathcal{B}$-structure has

$$\begin{tikzcd} B_A \times B_C \arrow[r, "\mu_{A,C}"] \arrow[d, "f_A \times f_C"'] & B_{A+C} \arrow[d, "f_{A+C}"] \\ BO(A) \times BO(C) \arrow[r, "w''_{A,C}"] & BO(A+C) \end{tikzcd}$$

μ is unital, associative.

4. For two (B, f)-structure $\mathcal{B}$ and $\mathcal{B}'$, suppose $\mathbb{C} \cap \mathbb{C}'$

is also a cofinal subset. Then a $(\mathcal{B}, f)$-map g is of course a natural transformation from $\mathfrak{B}$ to $\mathfrak{B}'$

$$\begin{tikzcd} B_A \arrow[r, "\lambda_{A,C}"] \arrow[d, "g(A)"'] & B_C \arrow[d, "g(C)"] \\ B'_A \arrow[r, "\lambda'_{A,C}"] & B'_C \end{tikzcd}$$

If $\mathfrak{B}$ and $\mathfrak{B}'$ are multiplicative, then we also require

$$\begin{tikzcd} B_A \times B_C \arrow[r, "\mu_{A,C}"'] \arrow[d, "g(A)\times g(C)"'] & B_{A+C} \arrow[d, "g(A+C)"] \\ B'_A \times B'_C \arrow[r, "\mu'_{A,C}"'] & B'_{A+C} \end{tikzcd}$$

If $\mathfrak{B}$ is a $(\mathcal{B}, f)$-structure, we require there is a

(B, f)- map $g : EO \to \mathcal{B}$.

Example : BO, EO

Example : For G a subgroup of O, and $G(A) \subseteq G(C)$,

$G(A) \times G(C) \longrightarrow G(A+C)$ for A, C orthagonal. Define

$$BG_A = EO(A)/G(A)$$

Then there is a canonical fibration

$f_A : BG_A = EO(A)/G(A) \longrightarrow BO(A) = EO(A)/O(A)$,

for each A.

$$\lambda_{A,C} : EO(A)/G(A) \longrightarrow EO(C)/G(C).$$

$$\mu_{A,C} : BG_A \times BG_C \longrightarrow BG_{A+C}.$$

$$g_A : EO(A) \longrightarrow EO(A)/G(A).$$

Example : BSO , $SO(A) \subseteq O(A)$.

Example : Consider $\mathbb{R}^\infty$ with basis $\{b_1, b_2, \cdots\}$ as the complex space $\mathbb{C}^\infty$ with basis $\{b_1, b_3, \cdots, b_{2n-1}, \cdots\}$ and $b_{2n} = i b_{2n-1}$.

P : a complex subspace of $\mathbb{C}^\infty$.

$U(P)$: $\mathbb{C}$- linear isometries from P to P.

$$U(P) \subseteq O(P).$$

Thus we can define $BU_P = EO(P)/U(P)$,

$P \in \mathfrak{P} :=$ { all complex subspaces of $\mathbb{C}^\infty$ } which is a cofinal set in $\mathfrak{U}$.

Example: $\mathbb{H}^\infty = \mathbb{H}\{b_1, b_5, \cdots, b_{4n+1} \cdots\}$.

$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$, with

$ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Note that $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$.

Let $\mathfrak{S}$ be all $\mathbb{H}$-spaces, which is a cofinal set of $\mathfrak{H}$.

For a $\mathbb{H}$ space Q, $Sp(Q) :=$ all $\mathbb{H}$-isometries.

$$\left\langle \sum_{j=0}^{t} x_j b_{4j+1}, \sum_{j=0}^{t} y_j b_{4j+1} \right\rangle = \sum_{j=0}^{t} x_j \overline{y_j},$$

where $\overline{a+bi+cj+dk} = a-bi-cj-dk$.

Then $Sp(Q) \subseteq U(Q) \subseteq O(Q)$.

Thus define $BSp_Q = EO(Q)/Sp(Q)$.

Remark : $EO \to BSp \to BU \to BSO \to BO$.

$\{1\} \subseteq Sp(Q) \subseteq U(Q)$ for Q an $\mathbb{H}$ space.

$U(P) \subseteq SO(P)$ for P an $\mathbb{C}$-space.

– Some properties.

$P \subseteq Q$ complex spaces, $\dim_{\mathbb{C}} Q = \dim_{\mathbb{C}} P + 1$, then

fibration $S^{2P+1} \longrightarrow BU(P) \xrightarrow{\lambda_{P,Q}} BU(Q)$

$P \subseteq Q$ $\mathbb{H}$-spaces, $\dim_{\mathbb{H}} Q = \dim_{\mathbb{H}} P + 1$, then

fibration $S^{4P+3} \longrightarrow BSp(P) \longrightarrow BSp(Q)$.

3. Manifolds with (B, f) – structure

A manifold with $\mathfrak{B}$ structure (M^n, e, g) consists

of **1.** A compact smooth manifold M^n.

2. An embedding $e: M^n \to A$, with A contains some

$C \in \mathfrak{C}$ and the dimension of $C^{\perp}$ in A is n.

3. A map $g: M^n \to B_C$, diagram commutes.

$$\begin{tikzcd}
 & B_C \arrow[d, "f_C"] \\
M^n \arrow[ur, "g"] \arrow[r, "\gamma"'] & BO(C)
\end{tikzcd}$$

where γ is the composition of $\gamma e|_{M^n}: M^n \to G_{A,C^\perp}$

and the canonical map $G_{A,C^\perp} \to BO(C)$.

We need to identify some $\mathfrak{R}$-structures on a manifold.

1. If $C' \subseteq A'$ and $C' \cong C$, $A' \cong A$, then we shall

identify (M^n, e, g) with (M^n, e', g')

2. If $C_0 \in \mathcal{C}$ and $C_0 \cap A = C$, then let $A_0 = A + C_0$.

then we identify (M^n, e, g) with (M^n, E, G)

$$E: M^n \xrightarrow{e} A \hookrightarrow A_0$$

$$G: M^n \xrightarrow{g} B_C \xrightarrow{\lambda_{C,C_0}} B_{C_0}.$$

4. If M^n has boundary, then we can choose $A = \mathbb{R}u + A'$ where $\mathbb{R}u$ orthogonal to A', and $C \subseteq A'$, such that

$$e(M^n, \partial M^n) \subseteq (A' + \mathbb{R}^+ u,\ A')$$

∂M^n inherits the $\mathcal{B}$-structure of M^n.

5. If $\mathcal{B}$ is multiplicative, $(M_1^{n_1}, e_1, g_1)$, $(M_2^{n_2}, e_2, n_2)$ will produce $(M_1^{n_1} \times M_2^{n_2}, e, g)$ with

$$e = e_1 \times e_2 : M_1^{n_1} \times M_2^{n_2} \longrightarrow A_1 + A_2,$$

$$g: M_1^{n_1} \times M_2^{n_2} \to B_{c_1} \times B_{c_2} \xrightarrow{\mu} B_{c_1+c_2}.$$

Example : EO - structure.

An EO-structure on (M^n, e, g) corresponds to a framing of the normal bundle ν_e of M^n.

$$
\begin{tikzcd}
 & V_{A,C^\perp} \arrow[r] \arrow[d] & V_{B,C^\perp} \arrow[r] \arrow[d, "\pi"] & EO(C) \arrow[d, "f_C"] \\
M^n \arrow[ur, "g''"] \arrow[urr, "g'"] \arrow[urrr, "g"] \arrow[r, "\nu_e|_{M^n}"'] & G_{A,C^\perp} \arrow[r, "i"'] \arrow[ur, phantom, "\circlearrowleft"] & G_{B,C^\perp} \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & BO(C)
\end{tikzcd}
$$

$\pi \circ g' = i \circ \nu_e|_{M^n}$.

$$J: N(e) \longrightarrow M^n \times \mathbb{R}^c$$

$$(e(m), v) \longmapsto (m, k_1, \cdots, k_c)$$

where $v \in \gamma_e(m)$, the normal space of point m,

$$g''(m) = \{b_1, b_2, \cdots, b_c\}. \quad v = k_1 b_1 + \cdots + k_c b_c.$$

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Example : BSO – structure

M^n is oriented $\iff$ $T(M^n)$ oriented

$\iff$ M^n has a BSO – structure.

$[\ H_n(B(m); \partial B(m)) \cong H_n(U_m, \partial U_m) \cong H_n(M^n, M^n - \{m\})\ ]$

Define $G^{SO}_{A,C^\perp} = \dfrac{O(A)}{SO(C) \times O(C^\perp)}$, oriented c-plane in A.

If $T(M^n)$ orientable, let $[B_m]$ be an orientation of $T_m M^n \subseteq$ A. Let $[B_m'']$ be the orientation of $N(e)_m$, such that

$$[B_m'] \cup [B_m''] = [C^\perp] \cup [C].$$

Thus we have such diagram

$$\begin{tikzcd} & G^{SO}_{A,C^\perp} \arrow[r] \arrow[d] & BSO(C) \arrow[d] \\ M^n \arrow[ur, "r^{SO}_e"] \arrow[r] & G_{A,C^\perp} \arrow[r] & BO(C) \end{tikzcd}$$

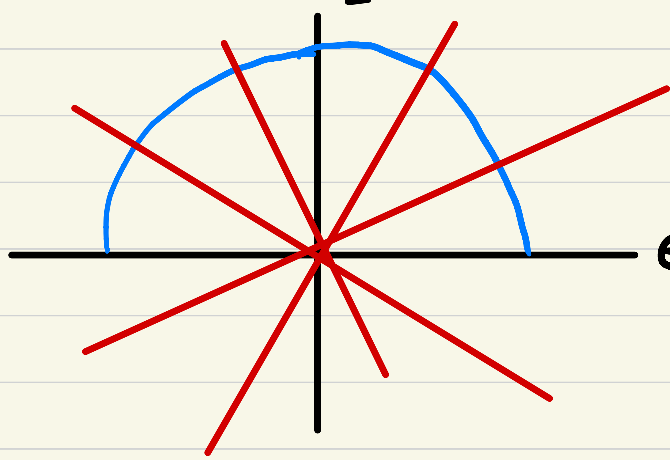
Example : BU-structure

Every complex manifold has a BU-structure . $\square$

4. Pontrjagin – Thom theorem and Thom spectra.

Let $\mathfrak{B}$ be a (B, f)-structure.

$$e_I : I \hookrightarrow \mathbb{R}e_1 \oplus \mathbb{R}e_2 \ , \quad e_I(t) = \cos(\pi t)e_1 + \sin(\pi t)e_2 .$$



$$I \longrightarrow G_{2,1} = \mathbb{R}P^1 \cong S^1 \longrightarrow BO(\mathbb{R}e_1)$$

Then we have a lift g,

$$\begin{tikzcd} & B_{\mathbb{R}e_1} \arrow[d] \\ I \arrow[ur, "g_I"] \arrow[r] & BO(\mathbb{R}e_1) \end{tikzcd}$$

that is (I, e_I, g_I) has $\mathfrak{B}$- structure.

For any $\mathfrak{B}$-manifold (M^n, e, g), there is

$$(M^n \times I,\ e \times e_I,\ \mu \circ (g \times g_I))$$

Define $-(M^n, e, g)$ to be the restriction on $t = 1 \in I$.

Define two $\mathfrak{B}$-manifolds (M^n, e, g), (N^n, f, h) are

<u>bordant</u> if there is a $\mathfrak{B}$-manifold (W^{n+1}, E, G) with

$$\partial (W^{n+1}, E, G) = (M^n, e, g) \sqcup -(N^n, f, h).$$

$\Omega_n^{\mathfrak{B}}$: equivalent class of n-dim $\mathfrak{B}$ manifolds

$(\Omega_n^{\mathfrak{B}}, \sqcup)$ is an abelian group :

Zero element $[\phi] \sim [B]$, B is the boundary of some $\mathfrak{B}$-manifolds.

Inverse : $-[(M^n, e, g)] = [-(M^n, e, g)]$,

since they together bound $(M^n \times I, e \times e_I, \mu(g \times g_I))$.

$(\Omega_*^{\mathfrak{B}}, \sqcup, \times)$ — a graded ring.

Example : Ω_*^{BO} is a $\mathbb{Z}/2\mathbb{Z}$ - algebra.

$$\Omega_0^{EO} = \Omega_0^{fr} = \mathbb{Z}$$

For a single point, there are two framings.

Namely $(\{*\}, F)$, $(\{*\}, -F) = -(\{*\}, F)$. And

$$[\{*\}, F] \neq [\{*\}, -F].$$

$$\Omega_1^{fr} = \mathbb{Z}/2\mathbb{Z}.$$

There is only one compact 1-dim manifold, S^1.

A framing is an element in $\pi_1(O(n))$,

$$\pi_1(O(n)) = \pi_1(SO(n)) \cong \pi_1(\mathbb{R}P^3) \text{ for } n \geq 3.$$

Thus there are two framings over S^1.

1. The trivial one bounds D^2.

2. (S^1, F) not bounds, but $-(S^1, F) = (S^1, F)$.

— Thom Spectra

$\pi: E \to B$ a vector bundle,

$D(\pi)$ — disk bundle, $S(\pi)$ — spherebundle

$M(\pi) := D(\pi)/S(\pi)$.

Suppose we have $\mathfrak{B}$-manifold (M^n, e, g),

$$\begin{tikzcd} & B_c \arrow[d, "f_c"] \\ M^n \arrow[ur, "g"] \arrow[r, "\nu_e"'] & BO(c) \end{tikzcd}$$

Thus $\nu_e = g^* f_c^* (P_c)$, P_c. the universal

C - vector bundle over $BO(C)$.

Define $\pi_C^{\mathfrak{B}} := f_C^*(P_C)$.

$$g : \nu_e \longrightarrow \pi_C^{\mathfrak{B}}$$

We can associate a map $\xi(A, C)$ to (M^n, e, g)

$$\xi : A^* \longrightarrow N_\varepsilon(e)/\partial N_\varepsilon(e) \to M\nu_e \xrightarrow{Mg} M\mathfrak{B}_C ,$$

where $M\mathfrak{B}_C = M\pi_C^{\mathfrak{B}}$.

Suppose $U \subseteq V$, $U^\perp$ complement in V.

$$\begin{tikzcd} B_U \arrow[r, "\lambda_{U,V}"] \arrow[d, "f_U"'] & B_V \arrow[d, "f_V"] \\ BO(U) \arrow[r, "j''_{U,V}"'] & BO(V) \end{tikzcd}$$

$$\lambda^*_{U,V} \pi^{\otimes}_V = \lambda^*_{U,V} f^*_V (P_V)$$

$$= f_U^* j''_{U,V} (P_V)$$

$$= f_U^* (\theta_{U^\perp} \oplus P_U)$$

$$= \theta_{U^\perp} \oplus \pi^{\otimes}_U$$

Thus we define

$$M\lambda_{U,V} : \quad U^{\perp *} \wedge M\mathcal{B}_U = M(\theta_{U^\perp} \oplus \pi_U^{\mathcal{B}})$$

$$\longrightarrow M(\pi_V^{\mathcal{B}}) = M\mathcal{B}_V$$

The collection of all such spaces $M\mathcal{B}_U$ and these structure maps is called the Thom spectrum $M\mathcal{B}$.

$$\pi_n(M\mathcal{B}) := \varinjlim_{U \subseteq V} [V^*, M\mathcal{B}_U] .$$

The direct system is taken over all (U, V) with $W = U^\perp$ in V, $\dim W = n$. And if $V \subseteq V_1$, then

$$[V^*, M\mathcal{B}_u] \longrightarrow [V_1^*, V^{\perp *} \wedge M\mathcal{B}_u]$$

$$\longrightarrow [V_1^*, M\mathcal{B}_{u_1}]$$

Moreover if $\mathcal{B}$ is multiplicative, then

$$\mu_{A,C} : B_A \times B_C \longrightarrow B_{A+C}$$

induces $M\mu_{A,C} : M\mathcal{B}_A \wedge M\mathcal{B}_C \longrightarrow M\mathcal{B}_{A+C}$,

and $\pi_n M\mathcal{B} \otimes \pi_{n'} M\mathcal{B} \to \pi_{n+n'} M\mathcal{B}$.

— The Pontrjagin – Thom Isomorphism.

$$\Omega_*^{\mathcal{B}} \longrightarrow \pi_* M\mathcal{B}$$

$$[M^n, e, g] \longmapsto \text{the image of } \xi \in [A^*, M\mathcal{B}_c] \text{ in the direct limit } \pi_n M\mathcal{B}$$

Let (W^{n+1}, E, G) be a bordism from

(M^n, e, g) to (N^n, f, h)

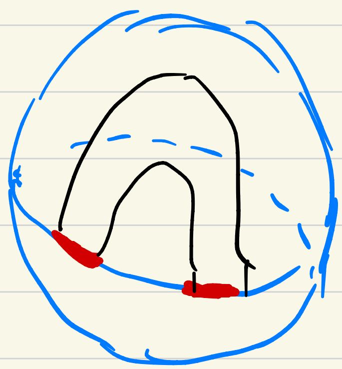
where $e : M^n \to A$, $f : N^n \to A$ and

$E : W^{n+1} \to A + \mathbb{R}^+ u$.

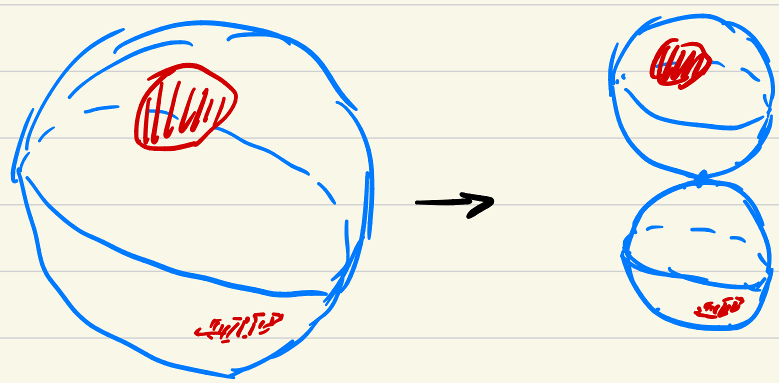
Then $\xi_{A,C}((M^n,e,g)\sqcup -(N^n,f,h)) : A^* \to M\mathfrak{B}_C$

But this map can be extended to an upper semisphere $(A+\mathbb{R}u)^*_+$. [Since we have map

$$\xi_{A+\mathbb{R}u^{+},\, C}(W^{n+1}) : (A+\mathbb{R}u)^*_+ \to M\mathfrak{B}_C \, .]$$



So its null-homotopic.



Corollary : $\Omega_*^{fr} = \pi_*^s(S^0)$

$$\Omega_n^{fr} = \Omega_n^{EO} = \pi_n MEO.$$

Note that $\pi_u^{EO} = f_u^*(P_n)$ is a vector bundle over $EO(U)$. Thus π_u^{EO} is trivial, $M(\pi_u^{EO}) \cong U^*$.

$$\pi_n MEO = \varinjlim_{U \subseteq V} [V^*, U^*]$$

$$= \varinjlim_{k} \pi_{n+k}(S^k)$$

$\Omega_0^{fr} = \mathbb{Z}$, $\Omega_1^{fr} = \mathbb{Z}/2\mathbb{Z}$.