Operads and the Recognition Principles

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Outline

1. Loop spaces and A_∞-structures
2. n-fold loop spaces and symmetric operads
3. Applications to algebraic K-theory
4. The monadic interpretation

Monoid structures

Definition

A monoid structure on a set X consists of a map

$$M\colon X \times X \to X$$

such that it satisfies associative law and has an identity element.

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for $k \geq 0$ such that

1. *$M(1)$ is the identity map,*
2. *the set $\{M(k)\}_{k\geq 0}$ is closed under multi-variable compositions.*

Homotopical monoid structures on loop spaces

A monoid structure on a set is "too rigid" in homotopy theory.

Example

Let Z be a based space. The space of based loops on Z is denoted by ΩZ. For each $r \in (0,1)$, we can define a multiplication

$$M_r : \Omega Z \times \Omega Z \to \Omega Z$$

such that $[0, r]$ encodes the first loop and $[r, 1]$ encodes the second loop. Similarly, given n disjoint subintervals of I, we can define a multiplication

$$(\Omega Z)^n \to \Omega Z$$

Note that any two choices of n disjoint subintervals of I will give homotopic multiplications.

Homotopical monoid structures on loop spaces

Let $\mathcal{A}(k)$ be the set that consists of sets of k disjoint subintervals of I. Note that we have an embedding

$$\mathcal{A}(k) \to \mathbb{R}^{2k}$$

by listing the $2k$ endpoints of the given k disjoint subintervals.

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Then we may view $\mathcal{A}(k)$ as a space and the previous construction determines

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for each $k \geq 0$.

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In other words, n-ary operations on ΩZ are governed by a space instead of a single map.

The embryo of operads

Proposition

If $Y = \Omega Z$ *for some* Z *then there is a family of subspaces* $\mathcal{M}(k) \subset \mathrm{Map}(Y^k, Y)$ *such that*

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2. *A collections of maps*

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The notion of non-symmetric operads

Definition

A **non-symmetric operad** $\mathcal{O}$ *is a collection of spaces* $\{\mathcal{O}(k)\}_{k\geq 0}$ *together with an element* $1 \in \mathcal{O}(1)$ *and maps*

$$\gamma\colon \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \mathcal{O}(k_k) \to \mathcal{O}(j_1 + \cdots + j_k)$$

for each choice of $k, j_1, \cdots, j_k \geq 0$ *such that*

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1. $\gamma(1, s) = s$ *and* $\gamma(s, 1, \ldots, 1) = s$ *for each* k *and* $s \in \mathcal{O}(k)$

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1. $\gamma(1, s) = s$ *and* $\gamma(s, 1, \ldots, 1) = s$ *for each* k *and* $s \in \mathcal{O}(k)$
2. *The collection is coherent under multi-variable compositions.*

The diagram of coherence of multi-variable compositions

$$
\begin{array}{ccc}
\mathcal{O}(k) \times \prod_{m=1}^{k}(\mathcal{O}(j_m) \times \prod_{n=1}^{j_m} \mathcal{O}(i_{mn})) & \xrightarrow{1\times\gamma} & \mathcal{O}(k) \times \prod_{m=1}^{k} \mathcal{O}(i_{m1} + \cdots i_{mj_m}) \\
\downarrow{=} & & \\
(\mathcal{O}(k) \times \prod_{m=1}^{k} \mathcal{O}(j_m)) \times \prod_{m,n} \mathcal{O}(i_{mn}) & & \downarrow{\gamma} \\
\downarrow{\gamma\times 1} & & \\
\mathcal{O}(j_1 + \cdots, j_k) \times \mathcal{O}(i_1 1) \times \cdots \times \mathcal{O}(i_{kj_k})) & \xrightarrow{\gamma} & \mathcal{O}(i_{11} + \cdots + i_{kj_k})
\end{array}
$$

Definition

A morphism between operads $\mathcal{O}$ and $\mathcal{O}'$ is a collection of continuous map $f_k : \mathcal{O}(k) \to \mathcal{O}'(k)$ such that they form functors between the above type of diagrams.

Example

The configuration spaces of subintervals of I we described before form **little interval non-symmetric operad** *denoted by $\mathcal{C}_1$.*

Examples of non-symmetric operads

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Example

If Y is any space, then the collection

$$\{\mathrm{Map}(Y^k, Y)\}$$

form a non-symmetric operad called the **endomorphism operad** *of Y and is denoted by $\mathcal{E}\mathrm{nd}_Y$.*

The action of operads on spaces

Definition

Let $\mathcal{O}$ be a non-symmetric operad and let Y be a space. An action of $\mathcal{O}$ on Y is a morphism between operads

$$\theta\colon \mathcal{O}(k) \to \mathcal{E}\mathrm{nd}_Y$$

More precisely, by adjunction, it is to assign

$$\theta\colon \mathcal{O}(k) \times Y^k \to Y$$

for each $k \geq 0$ with coherent conditions.

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Definition

The action of $\mathcal{O}$ on Y is **group-like** *if the monoid $\pi_0 Y$ is a group. We say that Y is an $\mathcal{O}$-***space.**

The recognition principle for A_∞-spaces

Definition

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Theorem (Recognition principle)

Y is weakly equivalent to ΩZ for some space Z if and only if Y has a group-like action of an A_∞ operad.

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Observations on higher homotopy groups

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4. $\pi_n(X, *) = \pi_0(\Omega^n X, *)$ has higher commutativity for $n > 2$?

$$X, \Omega X, \Omega^2 X, \Omega^3 X, \cdots, \Omega^n X, \cdots$$

The level of commutativity increases as n increases intuitively, but why? How should we describe this phenomenon precisely?

Observation: Why higher homotopy groups are always commutative

Given $[f], [g] \in \pi_2(X, *)$, i.e. $f, g \colon S^2 \to X$, the group operation on $\pi_2(X, *)$ is defined by

$$S^2 \xrightarrow{c} S^2 \vee S^2 \xrightarrow{f \vee g} X$$

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It is homotopy to the identity because we have an extra dimension to move the cubes, while we do not have such a space to move intervals in the dimension-1 case.

Symmetric operads

Recall that little intervals operads control ΩZ, what kinds of operads will control $\Omega^n Z$?

Definition

An operad is a (symmetric) operad $\mathcal{O}$ together with, for each k, there is a right action of Σ_k on $\mathcal{O}(k)$ and the coherent diagram is Σ-equivariant.

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Definition

Given an operad $\mathcal{O}$ and a space Y, the action of operad $\mathcal{O}$ on Y is an equivariant morphism $\mathcal{O} \to \mathcal{E}\mathrm{nd}_Y$. More precisely, the morphism

$$\mathcal{O}(k) \times Y^k \to Y$$

factors through $\mathcal{O}(k) \times_{\Sigma_k} Y^k$ for each k.

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1. *Define* $\mathcal{M}(j) = \Sigma_j$ *for* $j \geq 1$ *with* Σ_j*-right adjoint action and* $\mathcal{M}(0)$ *contain the single element* e_0*. The structure maps are given by multi-wreath products.*

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2. *Define* $\mathcal{N}(j) = \{f_j\}$ *a single point with trivial* Σ*-action.*

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2. *Define $\mathcal{N}(j) = \{f_j\}$ a single point with trivial Σ-action.*

Definition

*An A_∞-**operad** $\mathcal{O}$ is a Σ-free operad such that there exists a local Σ-equivalence $\mathcal{O} \to \mathcal{M}$ i.e. a morphism of non-symmetric morphism such that $\mathcal{O}(k) \to \Sigma(k)$ is an Σ_k-equivariant weak homotopy equivalence for each k.*
An E_∞ operad is a Σ-free operad such that each $\mathcal{O}(j)$ is contractible.

The little n-cubes operads

Definition

A **TD-map** $f : I^n \to I^n$ *is a composition* $T \circ D$*, where* T *is a translation and* D *is a dilation (i.e. multiplication by scalars). More precisely,* $f = f_1 \times \cdots \times f_j$*, where* $f_i : I \to I$ *is a linear function* $t \mapsto (y_i - x_i)t + x_i$ *for some* $0 \leq x_i < y_i < 1$*.*

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Definition

Given $n \geq 0$*, we let* $\mathcal{C}_n(k)$ *be the space consisting of k-tuple* $(j_1, \cdots, j_k)$ *of TD-maps with disjoint images, which is a subset of* $\mathrm{Map}(\sqcup_k I^n \to I^n)$ *and inherits the compact-one topology. The collection* $\{\mathcal{C}_n(k)\}_{k\geq 0}$ *forms an operad called* **the little n-cubes operad***, whose* Σ*-action structure maps are given evidently.*

The action of little n-cubes operads on loop spaces

Theorem

Given any space X, $\Omega^n X$ is a $\mathcal{C}_n$-space.

We define $\theta_{n,j} \colon \mathcal{C}_n(j) \times (\Omega^n X)^j \to \Omega X$ as follows

$$\theta_{n,j}(c, y)(v) = \begin{cases} y_r(u) & \text{if } c_r(u) = v \\ * & \text{if } v \notin \operatorname{im} c \end{cases}$$

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Remark

If $X = \Omega X'$, then $\theta_n = \theta_{n+1} \circ \sigma_n$, where $\sigma_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$ is given by $c_i \mapsto c_i \times \mathrm{id}$ and θ_{n+1} is the action on $\Omega^{n+1} X'$.

Computations on the little n-cube operads

Definition

Let M be an n-dimensional manifold. The j-th **configuration space** *$F(M;j)$ of M is defined to be*

$$\{(x_1, \cdots, x_j) \mid x_r \in M, x_r \neq x_s \text{ if } s \neq t\} \subset M^j$$

with subspace topology. Note that $F(M;j)$ is a jn-dimensional manifold and $F(M;j)$ with Σ_j-free right action.

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Theorem

We have a Σ_j-equivariant homotopy equivalence between $\mathcal{C}_n(j)$ and $F(\mathbb{R}^n;j)$

A map $g\colon \mathcal{C}_n(j) \to F(I^n;j)$ is defined by

$$g(c_1, \ldots, c_j) = (c_1(p), \cdots, c_j(p)), \text{ where } p = (\frac{1}{2}, \cdots, \frac{1}{2})$$

Computations on the configuration spaces

Theorem

Let M be an n-dimensional manifold $n \geq 2$. Let $Y_r \in F(M;r)$. We define

$$\begin{aligned} \pi_r : \quad F(M - Y_r; j - r) &\longrightarrow M - Y_r \\ (x_1, \ldots, x_{j-r}) &\longmapsto x_1 \end{aligned}$$

Then π_r is a fibration with fiber $F(M - Y_{r+1} - \{y_{r+1}\}; j - r - 1)$ over the point y_{r+1} and admits a cross-section if $r \geq 1$.

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Then π_r is a fibration with fiber $F(M - Y_{r+1} - \{y_{r+1}\}; j - r - 1)$ over the point y_{r+1} and admits a cross-section if $r \geq 1$.

Corollary

If $n \geq 3$, then $\pi_i F(\mathbb{R}^n; j) = \sum_{r=1}^{j-1} \pi_i(\vee^r S^{n-1})$; $\pi_i F(\mathbb{R}^2; j) = 0$ for $i \neq 1$ and $\pi_1 F(\mathbb{R}^2; j)$ is constructed from the free groups $\pi_1(\vee^r S^1)$.

Corollary

$\mathcal{C}_1$ is an A_∞ operad and $\mathcal{C}_n$ is a locally $(n-2)$-connected Σ-operad.

E_n-operads and the recognition principle for $\mathcal{C}_n$-spaces

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Remark

We can view $\mathcal{O}_n$ as the space of commutivity. Indeed, higher fold loop spaces will have better commutativity since their spaces of commutativity are more connected.

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Remark

We can view $\mathcal{O}_n$ as the space of commutivity. Indeed, higher fold loop spaces will have better commutativity since their spaces of commutativity are more connected.

Theorem (The recognition principle)

Y is weakly homotopy equivalent to $\Omega^n Z$ if and only if Y is a group-like E_n-space.

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Why Quillen's higher K-theory is a generalized cohomology theory

Roughly speaking, generalized cohomology theories = spectra via Brown's representability. Thus the problem is why Quillen's K-theory space is actually a spectrum.

Theorem

The category of infinite loop spaces is equivalent to the category connective spectra.

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Theorem

Suppose E is a symmetric monoidal category. Then the classifying space of E is an E_∞-space.

Group completions and the plus construction

Given a commutative ring R, Quillen's higher K-theory for $K(R)$ is built from the classifying space pf the category of finite projective R-modules. This space is denoted by $BGL(R)$ and it is an E_∞-space.

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Quillen's plus construction $BGL(R)^+$ is a kind of "higher group completion" on $BGL(R)$. The i-th K-group of R is defined to be $\pi_i(BGL(R)^+)$ and $BGL(R)^+$ is a group-like E_∞-space.

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The notion of monad

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A **monad** (C, μ, η) *in a category* $\mathcal{D}$ *consists of covariant functor* $C\colon \mathcal{D} \to \mathcal{D}$ *together with natural transformations of functors* $\mu\colon C^2 \to C$ *and* $\eta\colon \mathrm{id} \to C$ *such that some evident diagrams commute.*
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Given a monad C *on* $\mathcal{D}$, *a* C-**algebra** *is an object* $X \in \mathcal{D}$ *together with a structure map* $\xi\colon CX \to X$ *such that some evident diagrams commute.*

Operad=Operation+Monad

Given an operad $\mathcal{C}$. The associated monad (C, μ, η) is constructed by

$$CX = \coprod_{j \geq 0} \mathcal{C}(j) \times X^j / (\sim)$$

The relations consist of

1. The compatibility of $\sigma_i : \mathcal{C}(j) \to \mathcal{C}(j-1)$ and $s_i : X^{j-1} \to X^j$;

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Proposition

Given an operad $\mathcal{C}$ with associated monad C. Then the notion of a $\mathcal{C}$-space is equivalent to the notion of a C-algebra.

The insight of the recognition principle

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4. Passing to infinity by taking colimits, we have $\alpha_\infty \colon C_\infty X \to \Omega^\infty\Sigma^\infty X$.
5. Using techniques in cosimplicial spaces, such as two-side bar construction, we may have the delooping machinery from α_n and α_∞.