Sites, Sheaves, Formal Groups and Stacks

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Table of Contents

Grothendieck topology

Definition

A site is given by a category $\mathcal{C}$ and a class $\mathrm{Cov}(\mathcal{C}) \subset 2^{Mor(C)}$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$ where I is a small set, called coverings of $\mathcal{C}$, satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

(2) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$.

(3) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

Remark

In axiom (3) we require the existence of the fibre products $U_i \times_U V$ for $i \in I$. Actually almost all sites appear in algebraic geometry have any pullback.

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Example

(i)[Small Zariski site]
Let X be a topological space. Let X_{Zar} be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. Now define $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}\,(X_{Zar})$ if and only if $\bigcup U_i = U$.

(ii)[Big τ site]
Let Sch be the category of schemes, and $\tau \in \{Zar, et, Smooth, fppf, fpqc\}$. Let T be a scheme. An τ covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is (1)open immersion (2)étale (3)smooth (4)flat, locally of finite presentation (5)flat, respectively, and such that $T = \bigcup f_i\,(T_i)$. We denote the corresponding site to be Sch_τ. Appearently we have

$$\mathrm{Cov}(Zar) \subset \mathrm{Cov}(et) \subset \mathrm{Cov}(Smooth) \subset \mathrm{Cov}(fppf) \subset \mathrm{Cov}(fpqc)$$

.

Let $\mathcal{C}$ be a site.

Definition (Presheaf)

A presheaf of sets on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to Sets. Morphisms of presheaves are transformations of functors. The category of presheaves of sets is denoted $\mathrm{PSh}(\mathcal{C})$ or $Fun(C^{op}, Set)$. (Note $\mathcal{C}$ is not necessarily essentially small, so $\mathrm{PSh}(\mathcal{C})$ is not necessarily locally small)

Definition (Sheaf)

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a sheaf if for every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \overset{\mathrm{p}_0^*, \mathrm{p}_1^*}{\rightrightarrows} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of p_0^* and p_1^*.

Presheaves and Sheaves

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Definition (Sheafification)

Let $\mathcal{J}_U$ be the category of all coverings of U. The objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. Note that $\mathrm{Ob}\left(\mathcal{J}_U\right)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. We define

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{op}} H^0(\mathcal{U}, \mathcal{F})$$

where $H^0(\mathcal{U}, \mathcal{F}) = \left\{(s_i)_{i\in I} \in \prod_i \mathcal{F}\left(U_i\right), s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j} \forall i,j \in I\right\}$. We can verify $\mathcal{F}^+$ is separated and $s\mathcal{F} = (\mathcal{F}^+)^+$ is a sheaf. We call $s\mathcal{F}$ by the sheafification.

Warning: $\mathcal{J}_U$ is not necessarily a (essentially) small catgory, so not any presheaf on any site can be sheafificated. **Actually, there exists a presheaf on Sch_{fpqc} which admits no fpqc sheafification!**
However if we remove $fpqc$ and consider $\tau \in \{Zar, et, Smooth, fppf\}$, then all $\mathcal{J}_U$ in Sch_τ are essentially small and any presheaf in it can be sheafificated. **In the following context, we only consider the site whose $\mathcal{J}_U$ is essentially small and has any pullback.**

Sheafification

Proposition (Adjoint)

$PSh(\mathcal{C}) \rightleftarrows Sh(\mathcal{C})$ *is a pair of adjunction.*

Proposition

The sheafification functor $s : PSh(\mathcal{C}) \to Sh(\mathcal{C})$ *preserves any finite limit because the sheafification can be witten as a filtered colimit of underlying sets.*

Proposition (monomorphisms and epimorphisms)

Let $\varphi : \mathcal{F} \to \mathcal{G}$ *be a map of sheaves of sets or abelian groups, then*
(1) φ *is monomorphism iff for every object* U *of* $\mathcal{C}$ *the map* $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ *is injective.*
(2) φ *is epimorphism iff for every object* U *of* $\mathcal{C}$ *and every section* $s \in \mathcal{G}(U)$ *there exists a covering* $\{U_i \to U\}$ *such that for all* i *the restriction* $s|_{U_i}$ *is in the image of* $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

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Abelian sheaves

Proposition (Adjoint)

$PAbSh(\mathcal{C}) \rightleftarrows AbSh(\mathcal{C})$ *is still a pair of adjunction.*

Proposition

$PAbSh(\mathcal{C})$ *and* $AbSh(\mathcal{C})$ *are abelian categories.*

Remark

By the Yoneda lemma, if a presheaf of abelian groups is representable by an object H, then H admits a natural abelian group structure.

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Sheaves on the overcategory

Theorem

Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor $j_U : \mathcal{C}/U \longrightarrow \mathcal{C}$. Then we have the following equivalence of categories

$$Sh(\mathcal{C}/U) \rightleftarrows Sh(\mathcal{C})_{\downarrow U}$$

Remark

(1) In algebraic geometry, this equivalence tells us $Sh(Sch_{/S})_\tau$ is exactly the overcategory $Sh(Sch)_\tau \downarrow h_S$.

(2) This equivalence still holds even if we replace U by any sheaf $\mathcal{F}$.

$$Sh(\mathcal{C}/\mathcal{F}) \rightleftarrows Sh(\mathcal{C})_{\downarrow \mathcal{F}}$$

FPPF sheaves

Now let us focus on the big fppf site Sch_{fppf}.

Theorem

Let S be a base scheme, X be an S-scheme, then the representable functor $Hom_S(-, X)$ is an fppf sheaf on $Sch_{/S}$.

Theorem

For any $\tau \in \{Zar, et, Smooth, fppf\}$ (remove fpqc), $Aff \to Sch$ induces a natural equivalence of topoi

$$Sh(Sch)_\tau \xrightarrow{\sim} Sh(Aff)_\tau$$

A τ-sheaf is determined by its values on affine schemes!

Corollary

Note that any object in Aff_τ is compact, so the sheaf condition in it is a finite limit! So we get: In $Sh(Aff)_\tau$ any filtered colimit can be created in presheaf level, which commutes with any finite limit.

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Infinitesimal neighborhood along a subsheaf

Definition

Let $Y \subset X$ is an monomorphism of fppf sheaves on $Sch_{/S}$. We define $Inf_Y^k(X) \subset X$ to be the subsheaf whose value on an S-scheme T are given as follows: for a $t \in X(T)$, $t \in Inf_Y^k(X)(T)$ iff there is an fppf covering $\{T_i \to T\}$ and for each T_i associates a closed subscheme T_i' defined by an ideal whose $k+1$ power is (0) with the property that $t_{T_i'} \in X(T_i')$ is contained in $Y(T_i')$.

Example

If X and Y are S-schemes and $Y \to U \subset X$ is an immersion, then $Inf_Y^k(X) = Inf_Y^k(U) \simeq \mathrm{Spec}(\mathcal{O}_U/\mathcal{I}^{k+1})$ where $\mathcal{I}$ is the quasi-coherent ideal.

Proposition

Let $Z \subset X$ be a closed immersion of S-schemes with corresponding quasi-coherent ideal $\mathcal{I}$, then the value of $\hat{X}_Z = \varinjlim_k Inf_Z^k(X)$ on a S-scheme T equals to $\{t \in X(T) | t^*(\mathcal{I})$ is locally nilpotent$\}$.

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Completion of an fppf sheaf along a base point

We most consider the case when Y is a given base point, i.e. $Y(T) = \{*\} = h_S(T)$ for any S-scheme T. In this case we get a functor $\widehat{(-)} : Sh(Sch_{/S})^* \to Sh(Sch_{/S})^*$ by $(X, e) \mapsto (\varinjlim_k Inf_e^k(X), e)$. It is easy to check we have a natural inclusion $\hat{X} \subset X$, and that $\widehat{\widehat{X}} \subset \widehat{X}$ is a natural isomorphism.

We say an $X \in Sh(Sch_{/S})^*$ is complete iff $\hat{X} = X$.

Theorem

(a) $(\hat{-})$ *preserves finite limits, so* $CSh(Sch_{/S})^*$ *has finite limits, which are created in* $Sh(Sch_{/S})^*$.

(b) $Forget : CSh(Sch_{/S})^* \rightleftarrows Sh(Sch_{/S})^*$ *is an adjoint pair.*

(c) $Forget : CAbSh(Sch_{/S}) \rightleftarrows AbSh(Sch_{/S})$ *is an adjoint pair.*

Formal geometry

We have known that the equivalence of topoi $Sh(Sch)_{fppf} \longrightarrow Sh(Aff)_{fppf}$, so we will be free to exchange things from each other.

Definition

The $\hat{\chi}$ is a full subcategory of $Fun(Rings, Sets)$ which consists of functors $X : Rings \to Sets$ that is a small filtered colimit of corepresentable functors. More precisely, there must be a small filtered category $\mathcal{J}$ and a functor $i \mapsto X_i = Hom(R_i, -)$ such that $X = \varinjlim_i X_i$.

Actually $\hat{\chi}$ is the category of “formal schemes" in Strickland's paper, which equals to $(Pro - Ring)^{op}$ or $Ind - Aff$.

We donote $LRing$ and $FRing$ to be the category of linearly topological rings and complete linearly topological rings respectively. Then we have fully faithful embeddings

$$FRing \to \hat{\chi} \to Sh(Aff)_{fppf}$$

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Definition

Let $X \in CSh(Sch_{/S})^*$, we call it a (pointed) formal Lie variety iff zariski locally on S, the F is isomorphic to $Spf(\mathcal{O}_S[[x_1, ..., x_N]])$ as fppf sheaves for some $N \geq 0$.

Theorem

Let $X \in CSh(Sch_{/S})^*$, TFAE

(1) X is a formal Lie variety.

(2) Zariski locally on S, the X is isomorphic to $Spf(\mathcal{O}_S[[x_1, ..., x_N]])$ as pointed sheaves for some $N \geq 0$.

(3) (a) The $\mathrm{Inf}^k(X)$ is representable for all $k \geq 0$.

(b) The $\omega_X = e^*(\Omega_{\mathrm{Inf}^1(X)/S}) = e^*(\Omega_{\mathrm{Inf}^k(X)/S})$ is a finite locally free sheaf on S.

(c) Denoting by $gr_*^{inf}(X)$ the graded $\mathcal{O}_S$-algebra $\bigoplus_{k \geq 0} \mathcal{I}_k^k$, such that $gr_i^{\mathrm{inf}}(X) = gr_i(\mathrm{Inf}^i(X))$ holds for all $i \geq 0$. We have an isomorphism $Sym_S(\omega_X)_* \xrightarrow{\sim} gr_*^{inf}(X)$ induced by the canonical mapping $\omega_X \xrightarrow{\sim} gr_1^{\mathrm{inf}}(X)$.

Formal Lie varieties

Theorem

Let X be a smooth S-scheme with a base point $e \in X(S)$, then $\hat{X}$ is a formal Lie variety.

Theorem

Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety. If $S = \mathrm{Spec}(R)$ is affine, then we have a (non-canonical) isomorphism $X \to Spf(\widehat{Sym}_S(\omega_X))$ as pointed sheaves.

The second theorem is based on the fact that a finite locally free sheaf is a projective object in $Qcoh(S)$ if S is affine, whic tells us any formal Lie variety on an affine base S is from the completion of a pointed smooth S-scheme.

Corollary

Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety (S here is not assumed to be affine), then X is a formally smooth fppf sheaf, which means $X(Spec(A)) \to X(Spec(A/I))$ is surjective for any $A \to A/I$ over S with a square-zero ideal I.

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Formal Lie groups

Definition

A formal Lie group is an Abelian sheaf $X \in AbSh(Sch_{/S})$ whose underlying pointed sheaf is a formal Lie variety.

We more care about 1-dim formal Lie groups, in most references they are directly called by the “formal group". Now we will show that formal groups are equivalent to graded formal group laws on even weakly periodic graded rings.

Definition (EWP)

A graded ring R_* is even weakly periodic iff it satisfies following conditions
(a) $R_2 \otimes_{R_0} R_{-2} \to R_0$ is isomorphic;
(b) $R_1 = 0$.

Proposition

From the definition, for an EWP R_* we immediately get
(1) $R_2 \otimes_{R_0} R_n \to R_{n+2}$ *is isomorphic for any* $n \in \mathbb{Z}$. *(2)*$R_{odd} = 0$.
(3) $R_2 \in Pic(R_0)$ *with* $(R_2)^{-1} = R_{-2}$.

Formal groups

Now let us calculate the data of a formal group.

Proposition

$Hom_{Sh(S)^*}(Spf(\widehat{Sym}_S(M)), Spf(\widehat{Sym}_S(N))) = \prod_{i=1}^{+\infty} Hom_{\mathcal{O}_S-Mod}(N, Sym_i(M))$ where $M, N \in Qcoh(S)$.

Corollary

Let $X \in CSh(Sch_{/S})^*$ be a formal Lie variety over an affine base $S = \mathrm{Spec}(R)$, then $Hom_{Sh(S)^*}(X \times X, X) \simeq \prod_{(i,j)|i+j \geq 1} Hom_{\mathcal{O}_S-Mod}(\omega_X, \omega_X^{i+j}) = \prod_{(i,j)|i+j\geq 1} \omega_X^{i+j-1}$.
If it satisfies the associated law then it coincides with a graded formal group law on $Sym(\omega_X)_*$ or the EWP $Sym^{\pm}(\omega_X)_* = \bigoplus_{i \in \mathbb{Z}} \omega_X^i$.

Theorem

The construction above actually gives an equivalence of moduli stacks $\mathcal{M}_{FG} \xrightarrow{\sim} \mathcal{M}_{FGL_s(EWP)}$ over Aff.

Fibered category(Cartesian fibration)

Definition

Let $F : E \to C$ be a functor. An arrow $\phi : \xi \to \eta$ of F is cartesian if for any arrow $\psi : \zeta \to \eta$ in F and any arrow $h : \mathrm{p}_F\zeta \to \mathrm{p}_F\xi$ in $\mathcal{C}$ with $\mathrm{p}_F\phi \circ h = \mathrm{p}_F\psi$, there exists a unique arrow $\theta : \zeta \to \xi$ with $\mathrm{p}_F\theta = h$ and $\phi \circ \theta = \psi$, as in the commutative diagram
If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we also say that ξ is a pullback of η to U.