Formal Group Laws and Formal Groups

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Formal Group Law

Definition (Formal Group Law)

Let R be a commutative ring. A (one-dimensional commutative) formal group law over R is a power series $f(x,y) \in R[[x,y]]$ satisfying three conditions:

$$f(x,0) = f(0,x) = x$$

$$f(x,y) = f(y,x)$$

$$f(x,(f(y,z))) = f(f(x,y),z)$$

Let $\mathrm{FGL}(R) := \{\text{formal group laws over } R\} \subset R[[x,y]]$.

$\mathrm{FGL}\colon \mathit{Rings} \to \mathit{Sets}$ is a covariant functor.

An Example of Formal Group Law

Let K be a local field with integer ring $\mathcal{O}_K$. Suppose the residue field of $\mathcal{O}_K$ has $q = p^k$ elements for some prime number p and integer k.

Fix an uniformizer π of $\mathcal{O}_K$.

$$\mathscr{F}_\pi := \left\{ f \in t\mathcal{O}_K[[t]] : \ f \equiv pt \pmod{t^2},\ f \equiv t^q \pmod{\pi} \right\}$$

Proposition

For any $f \in \mathscr{F}_\pi$, there exists a unique $F \in FGL(\mathcal{O}_K)$, such that $f \in End(F)$.

It can be shown that different choices of π, f will give isomorphic F.

Lazard Ring

For any $f(x,y) = \sum c_{i,j}x^i y^j \in R[[x,y]]$,

$f \in \mathsf{FGL}(R) \Leftrightarrow c_{i,j}$ satisfy certain polynomial equations

given by the previous conditions on formal group laws.

Let $L = \mathbb{Z}[t_{i,j}]/Q$, where $Q \subset \mathbb{Z}[t_{i,j}]$ is the ideal generated by the polynomial equations mentioned above.

Then $f_L(x,y) = \sum t_{i,j}x^i y^j \in \mathsf{FGL}(L)$.

Lazard Ring

For each commutative ring R and $f(x,y) = \sum c_{i,j} x^i y^j$,

$$\begin{array}{ccccccc} L & \longrightarrow & R & \rightsquigarrow & \mathsf{FGL}(L) & \longrightarrow & \mathsf{FGL}(R) \\ t_{i,j} & \longmapsto & c_{i,j} & & f_L & \longmapsto & f \end{array}$$

We get a surjective map:

$$\mathsf{Hom}(L,R) \to \mathsf{FGL}(R)$$

Easy to see that this is bijective $\Rightarrow$ $\mathsf{FGL} \cong \mathsf{Hom}(L,-)$ representable.

L is called the Lazard ring.

Lazard's Theorem

Let $\deg t_{i,j} = 2(i+j-1)$.

If we let $\deg x = \deg y = -2$, $f(x,y) = \sum t_{i,j}x^iy^j$ has degree -2.

$\Rightarrow$ $f\big(f(x,y),z\big), f\big(x,f(y,z)\big)$ also have degree -2.

$\Rightarrow$ coefficients of $x^iy^jz^k$ in $f\big(f(x,y),z\big), f\big(x,f(y,z)\big)$ have degree $2(i+j+k)-2$.

$\Rightarrow$ Q is a homogeneous ideal, and it gives a grading on L.

Theorem (Lazard)

$L \cong \mathbb{Z}[t_1, t_2, \cdots]$, *where* $\deg t_i = 2i$.

Lazard's Theorem: Sketch Proof

Let $g(x) = x + t_1x^2 + t_2x^3 + \cdots \in \mathbb{Z}[t_1, t_2, \cdots][[x]]$

$$g\big(g^{-1}(x) + g^{-1}(y)\big) \in \mathsf{FGL}\big(\mathbb{Z}[t_1, t_2, \cdots]\big) \rightsquigarrow \phi\colon L \to \mathbb{Z}[t_1, t_2, \cdots]$$

Let $I = L_{>0}$, $J = \mathbb{Z}[t_1, t_2, \cdots]_{>0}$ consisting of elements of positive degree. Then $(I/I^2)_{2n}$ and $(J/J^2)_{2n}$ inherit grading from $L, \mathbb{Z}[t_1, t_2, \cdots]$.

Lazard's Theorem: Sketch Proof

Lemma

For any $n \in \mathbb{Z}_{>0}$, $\phi \colon L \to \mathbb{Z}[t_1, t_2, \cdots]$ induces an injection

$$(I/I^2)_{2n} \to (J/J^2)_{2n} \cong \mathbb{Z}$$

$$im\big((I/I^2)_{2n} \to (J/J^2)_{2n}\big) = \begin{cases} p\mathbb{Z} & n+1 = p^f \text{ for some } f \in \mathbb{Z}_{>0} \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

Then we have $(I/I^2)_{2n} \cong \mathbb{Z}$.

Choose $t_n \in I_{2n} = L_{2n}$ lifting generators for $(I/I^2)_{2n} \cong \mathbb{Z}$.

$\rightsquigarrow \theta \colon \mathbb{Z}[t_1, t_2, \cdots] \to L$. It can be shown that θ is an isomorphism.

Lazard's Theorem

ϕ is injective but not surjective, but it is an iso after tensoring with $\mathbb{Q}$.

Proposition

Let R be a $\mathbb{Q}$-algebra. For any $f \in FGL(R)$, there exists

$g(x) = x + a_1 x + a_2 x^3 + \cdots$, *s.t.* $f(x,y) = g\big(g^{-1}(x) + g^{-1}(y)\big)$.

But this is not true for R when $\mathrm{char}(R) > 0$.

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Height

Proposition

Let R be a commutative ring with char$(R) = p$ and $f \in$ FGL(R). Then either $[p]_f(t) = 0$ or $[p]_f(t) = \lambda t^{p^n} + O(t^{p^n+1})$ for some $n > 0$ and $\lambda \neq 0$.

Definition

Let R be a commutative ring and $f \in \mathrm{FGL}(R)$. Fix a prime number p.

Let v_n denote the coefficient of t^{p^n} in $[p]_f$.

Say f has height $\geqslant n$ if $v_i = 0$ for $i < n$

f has height exactly n if f has height $\geqslant n$ and $v_n \in R$ invertible.

Height

Let $f, f' \in \mathrm{FGL}(R)$ and g is an iso between f, f'.

$$[p]_{f'}(x) = (g \circ [p]_f \circ g^{-1})(x) \rightsquigarrow \mathrm{ht}(f) = \mathrm{ht}(f')$$

If $\mathrm{char}(R) = p$,

Let $f(x, y) = x + y + xy$. Then $[p]_f(x) = (1 + x)^p - 1 = x^p$, so $\mathrm{ht}(f) = 1$.

Let $g(x, y) = x + y$. Then $[p]_g(x) = 0$, so $\mathrm{ht}(g) = \infty$.

$\Rightarrow f \not\sim g$.

Classification of Formal Group Laws

Proposition

Let R be a commutative ring with char$(R) = p$ and $f \in$ FGL(R). TFAE:

(i) $ht(f) \geqslant n$.

(ii) There exists an $f' \sim f$ such that $f'(x,y) \equiv x + y \pmod{(x,y)^{p^n}}$.

Corollary

Let $f \in$ FGL(R). If $ht(f) = \infty$, $f \sim x + y$.

Classification of Formal Group Laws

Theorem

Let k be an algebraically closed field with char$(k) = p$. Then two formal group laws f, f' are isomorphic if and only if ht$(f) =$ ht(f').

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Formal Schemes

Definition

A linear topology on a ring R is a topology such that 0 has a fundamental system of neighborhoods consisting of ideals.

Let $LRings$ be the category of linearly topologized ring with continuous ring homomorphisms.

Given a linearly topologized ring R, let

$$\mathrm{Spf}(R) \colon LRings \to Sets$$

$$\mathrm{Spf}(R)(S) := LRings(R, S)$$

Formal Schemes

For an arbitrary ring R, R is linearly topologized with discrete topology, so $Rings$ is a full subcategory of $LRings$.

For $R \in LRings, S \in Rings$,

$$\mathrm{Spf}(R)(S) = LRings(R, S) = \mathrm{colim} Rings(R/J, S)$$

where J runs through all ideals in the fundamental system of neighborhoods of 0.

Formal Schemes

For complete local ring R, S with maximal ideals $\mathfrak{m}_R, \mathfrak{m}_S$,

linearly topologized by the $\mathfrak{m}_R$-adic, $\mathfrak{m}_S$-adic topology respectively.

$$\mathrm{Spf}(R)(S) = LRings(R, S)$$

is the local homomorphisms between R, S.

Formal Schemes

Example

Fix a commutative ring R and $R[[x_1, \cdots, x_n]]$ with $(x_1, \cdots, x_n)$-adic topology. Let $\widehat{\mathbb{A}}^n_R := \mathrm{Spf} R[[x_1, \cdots, x_n]]$

For an R-algebra T with discrete topology,

$$\widehat{\mathbb{A}}^n_R(T) = \mathrm{colimHom}_R(R[[x_1, \cdots, x_n]]/(x_1, \cdots, x_n)^k, T) = \mathrm{Nil}(T)^n$$

If R is a complete local ring and T is a complete local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{m}$-adic topology

$$\widehat{\mathbb{A}}^n_R(T) = LRings(R[[x_1, \cdots, x_n]], T) = \mathfrak{m}^n$$

Formal Groups

Definition (Formal Scheme)

A formal scheme is a functor of the form $\mathrm{Spf}(R)$.

Definition (Affine Formal Group)

An affine formal group over R of dimension n is a group object in formal schemes isomorphic to $\widehat{\mathbb{A}}_R^n$.

Definition (Formal Group)

A formal group is formal scheme that is locally an affine formal group.

Formal Groups

Given an affine formal group $\widehat{G}$ of dimension 1,choose a coordinate $\widehat{G} \overset{x}{\cong} \widehat{\mathbb{A}}^1$.

The addition laws on $\widehat{G}, \widehat{\mathbb{A}}^1$ give us

$$\begin{array}{ccc} \widehat{G} \times \widehat{G} & \xrightarrow{\sigma} & \widehat{G} \\ {\scriptstyle x\times x}\downarrow & & \downarrow{\scriptstyle x} \\ \widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1 & \longrightarrow & \widehat{\mathbb{A}}^1 \end{array}$$

Then $(x \times x)^{-1} \circ \sigma \circ x$ is a morphism between $\mathrm{Spf}(R[[x,y]]) \to \mathrm{Spf}(R[[t]])$, which is an element in $LRings(R[[t]], R[[x,y]])$ by Yoneda Lemma.

The image of t is an one-dimensional formal group law.