Rational Homotopy Theory.

- rational space : 1. simple connected space X , i.e. $\pi_1(X)=0$

  2. $\pi_*(X)\otimes_{\mathbb{Z}}\mathbb{Q} \cong \pi_*(X) \iff \tilde{H}_*(X;\mathbb{Z})\otimes\mathbb{Q} \cong \tilde{H}_*(X;\mathbb{Z}) \cong \tilde{H}_*(X;\mathbb{Q})$

the existence of rational space : constructure rational space.

simple connected space X.

$\downarrow$ CW- approximation.

CW - complex $\quad (X^n = X^{n-1} \sqcup_\alpha e_\alpha^n = X^{n-1} \cup_{f_\alpha} D_\alpha^n)$

$\downarrow$ $(D^{n+1}, S^n) \Rightarrow (D_{\mathbb{Q}}^{n+1}, S_{\mathbb{Q}}^n)$

rational space. $X_{\mathbb{Q}}$. $\quad (X^{(n)} = X^{(n-1)} \sqcup_\alpha e_{\mathbb{Q},\alpha}^n = X^{(n-1)} \cup_{F_\alpha} D_{\mathbb{Q},\alpha}^n)$

$S_{\mathbb{Q}}^n = \left(\bigvee_{i=0}^{\infty} S_i^n\right) \cup_h \left(\coprod_{j=1}^{\infty} D_j^{n+1}\right)$, where D_j^{n+1} is attached by a map $S^n \to S_{j-1}^n \vee S_j^n$, representing $[S_{j-1}^n] - k_j [S_j^n]$.

$X^n \subset X^{(n)} \subset X^{n+1}$

Thm: $\pi_*(X;\mathbb{Z})$ is $\mathbb{Q}$-module $\iff \tilde{H}_*(X;\mathbb{Z})$ is $\mathbb{Q}$-module $\iff \tilde{H}_*(\Omega X;\mathbb{Z})$ is $\mathbb{Q}$-module.

Def. rationalization: $\varphi : X \to X_{\mathbb{Q}}$ s.t. φ induces an isomorphism

$$\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \pi_*(X_{\mathbb{Q}})$$

Thm: $\varphi : X \to Y$ is a rationalization iff. $\pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_*(Y)$ and $H_*(\varphi; \mathbb{k})$ is an isomorphism.

Thm. space X is simplex connected, the rationalizations are unique up to homotopy equivalence rel X.

$f : X \to Y$. following conditions are equivalent:

① $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism

② $H_*(f; \mathbb{Q})$ is an isomorphism

③ $H^*(f; \mathbb{Q})$ is an isomorphism.

- Transform topological space into commutative cochain algebras.

$A_{PL} = \{(A_{PL})_n\}_{n \geq 0}$: $\quad \left(A_{DR}(\Delta^n) = C^\infty(\Delta^n) \otimes_{(A_{PL}^0)_n} (A_{PL})_n \right)$

simplicial commutative cochain algebra

1. $(A_{PL})_n = \dfrac{\Lambda(t_0, \cdots, t_n, y_0, \cdots, y_n)}{(\Sigma t_i - 1, \Sigma y_j)}$ $\quad dt_i = y_i, \ dy_j = 0$, $\quad |t_i| = 0, \ |y_j| = 1$.

2. face and degeneracy morphisms.

$\partial_i : (A_{PL})_{n+1} \to (A_{PL})_n \qquad s_j : (A_{PL})_n \to (A_{PL})_{n+1}$

$$\partial_i : t_k \mapsto \begin{cases} t_k & k < i \\ 0 & k = i \\ t_{k-1} & k > i \end{cases} \qquad s_j : t_k \mapsto \begin{cases} t_k & k < j \\ t_k + t_{k+1} & k = j \\ t_{k+1} & k > j \end{cases}$$

$A_{PL} = \{(A_{PL})_n\}_{n\geq 0}$, K be a simplicial set.

$A_{PL}(K) = \{A_{PL}^p(K)\}_{p\geq 0}$ is the "ordinary" cochain complex (algebra)

1. A_{PL}^p is the set of simplicial set morphisms from K to A_{PL}^p

   $\Phi \in A_{PL}^p(K)$ $\qquad \sigma \overset{\Phi}{\longmapsto} \Phi_\sigma$ s.t. $\Phi_{\partial_i \sigma} = \partial_i \Phi_\sigma$, $\Phi_{s_j \sigma} = s_j \Phi_\sigma$

2. $(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma$, $(\lambda \Psi)_\sigma = \lambda \cdot \Psi_\sigma$, $(d\Psi)_\sigma = d(\Psi_\sigma)$

3. If A_{PL} is a simplicial cochain algebra, then (general, consider A)

   $(\Phi \cdot \Psi)_\sigma = \Phi_\sigma \cdot \Psi_\sigma$

4. $\varphi : K \to L$ morphism of simplicial sets then

   $A_{PL}(\varphi) : A_{PL}(K) \longleftarrow A_{PL}(L)$

   is the morphism of cochain complexes (algebra) defined by.

   $(A_{PL}(\varphi)\Phi)_\sigma = \Phi_{\varphi\sigma}$

$A_{PL}(X) \triangleq A_{PL}(S_*(X))$ $\quad$ ($S_*(X)$: the set singular simplexes (simplices) on a space X)

The simplicial cochain algebra C_{PL} : $(C_{PL})_n \triangleq C^*(\Delta[n])$, $C_{PL} = \{(C_{PL})_n\}_{n\geq 0}$

($C^*(X;\Bbbk)$ is called the normalized singular chain complex of X)

Thm. 1. the natural morphisms of cochain algebras,

$$C_{PL}(K) \longrightarrow (C_{PL} \otimes A_{PL})(K) \longleftarrow A_{PL}(K) \qquad (K: \text{simplicial set})$$

are quasi-isomorphism.

2. there is a natural isomorphism $C_{PL}(K) \xrightarrow{\cong} C^*(K)$ of cochain algebras.

Cor. $\quad H^*(X) \cong H(A_{PL}(X))$

- Sullivan models

Def. Sullivan algebra: a commutative cochain algebra $(\Lambda V, d)$, satisfying

1. $V = \{V^p\}_{p \geq 1}$

2. $V = \bigcup_{k=0}^{\infty} V(k)$, $V(0) \subset V(1) \subset \cdots$

   $d = 0$ in $V(0)$ and $d: V(k) \to \Lambda V(k-1)$, $k \geq 1$.

2'. there exist graded subspaces $V_k \subset V(k)$, s.t. $\Lambda V(k) = \Lambda V(k-1) \otimes \Lambda V_k$.

   $d: V_k \to \Lambda V(k-1)$

(3. minimal: $\mathrm{Im}\, d \subset \Lambda^+ V \cdot \Lambda^+ V$)

Def. Sullivan model: $m: (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$

Prop. Any commutative cochain algebra (A, d), satisfying $H^0(A) = \Bbbk$, has a Sullivan model.

$$\left\{ \begin{array}{c} \text{rational homotopy} \\ \text{types} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{isomorphism classes of minimal} \\ \text{Sullivan algebras over } \mathbb{Q} \end{array} \right\}$$

(simple connected) $(V^1 = 0)$

Def. morphisms $\varphi_0, \varphi_1 : (\Lambda V, d) \to (A, d)$ are homotopic, if there is a morphism

$$\Phi : (\Lambda V, d) \to (A, d) \otimes (\Lambda(t, dt), d)$$

s.t. $(id \cdot \varepsilon_i) \Phi = \varphi_i$ $i = 0, 1$. ($\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \to \Bbbk$, by $\varepsilon_0(t) = 0$, $\varepsilon_1(t) = 1$)

Φ is called a homotopy from φ_0 to φ_1 ($\varphi_0 \sim \varphi_1$).

Sullivan representative for $f : X \to Y$: $\varphi : (\Lambda W, d) \to (\Lambda V, d)$

$$\begin{array}{ccc} A_{PL}(X) & \xleftarrow{A_{PL}(f)} & A_{PL}(Y) \\ {\scriptstyle m_X}\uparrow & \sim & \uparrow{\scriptstyle m_Y} \\ (\Lambda V, d) & \xleftarrow{\exists ! \varphi} & (\Lambda W, d) \end{array}$$

$$\left\{ \begin{array}{c} \text{homotopy classes of} \\ \text{maps } X \to Y \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{homotopy classes of} \\ \text{morphisms } (\Lambda W, d) \to (\Lambda V, d) \end{array} \right\}$$

Proof of the existence of Sullivan model: . . .

$m_0 : (\Lambda V_0, 0) \to (A, d)$ s.t. $H(m_0) : V_0 \xrightarrow{\cong} H^+(A)$

$\Downarrow$

$m_k : \left(\Lambda \left(\bigoplus_{i=0}^{k} V_i \right), d \right) \to (A, d)$

$\Downarrow$

$V_{k+1} \cong \operatorname{Ker} H(m_k)$, $[z_\alpha]$ is a basis for $\operatorname{Ker} H(m_k)$

$v_\alpha \in V_{k+1}$, $d v_\alpha = z_\alpha$

$m_k z_\alpha = d a_\alpha$, $a_\alpha \in A$, ~~$m_{k+1} d v_\alpha = d m_{k+1} v_\alpha$~~ $m_{k+1} v_\alpha = a_\alpha$.

. . .

$V = \bigoplus_i V_i$

Example 1. S^k.

if k is odd, $m: (\Lambda(e), 0) \xrightarrow{\simeq} A_{PL}(S^k)$

if k is even, $m: (\Lambda(e, e'), de' = e^2) \xrightarrow{\simeq} A_{PL}(S^k)$

Example 2. Product.

$m_X : (\Lambda V, d) \to A_{PL}(X)$ $\quad$ $m_Y : (\Lambda W, d) \to A_{PL}(Y)$

$$m_X \cdot m_Y : (\Lambda V, d) \otimes (\Lambda W, d) \xrightarrow{\simeq} A_{PL}(X \times Y)$$

Example 3. H-space have minimal Sullivan models of the form $(\Lambda V, 0)$

~~Example.~~

~~Lemma.~~

Example 4. Wedge.

$$m_\alpha : (\Lambda V_\alpha, d) \xrightarrow{\simeq} A_{PL}(X_\alpha)$$

$$\prod_\alpha (\Lambda V_\alpha, d) \xrightarrow{\simeq} \prod_\alpha A_{PL}(X_\alpha) \longleftarrow A_{PL}(\bigvee_\alpha X_\alpha)$$

$\Uparrow$

not sullivan algebra.

Example 5. $\underset{x}{S^3} \times \underset{y}{S^3} \times \underset{z}{S^5} \times \underset{a,u}{S^6}$

$$(\Lambda V, d) = \Lambda(x, y, z, a, u;\ dx = dy = dz = da = 0,\ du = a^2)$$

Adjunction space.

$f: Y \to X$. $\qquad \varphi: (\Lambda V_X, d) \to (\Lambda V_Y, d)$ is Sullivan representive of f.

$$\begin{array}{ccc}
(\Lambda V_Y, d) & \longleftarrow & \left(\mathbb{k} \oplus [B \otimes \Lambda^+(t, dt)], d\right) \\
\uparrow & & \uparrow \\
(\Lambda V_X, d) & \longleftarrow & \left(\Lambda V_X \times_{\Lambda V_Y} \left(\mathbb{k} \oplus [B \otimes \Lambda^+(t, dt)]\right), d\right)
\end{array}$$

$\nearrow$ a commutative model for $X \cup_f CY$

Relative Sullivan model. $\quad (B \otimes \Lambda V, d) \quad \ldots$ be similar to sullivan model.

$$\begin{cases}
(B, d) = (B \otimes 1, d) \qquad H^0(B) = \mathbb{k}. \\
1 \otimes V = V = \{V^p\}_{p \geq 1} \\
V = \bigcup_{k=0}^{\infty} V(k) \qquad V(0) \subset V(1) \subset \cdots \quad \text{s.t.} \\
\quad d: V(0) \to B \quad \text{and} \quad d: V(k) \to B \otimes \Lambda V(k-1) \quad k \geq 1
\end{cases}$$

given $\varphi: (B, d) \to (C, d)$, s.t. $H^0(B) = \mathbb{k}$, Sullivan model for φ is

$$m: (B \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d)$$

(minimal : $\operatorname{Im} d \subset B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$)

the existence of minimal : $(B \otimes \Lambda V, d)$

idea : $V \Rightarrow W \oplus U \oplus d_0 U$.

$$\begin{array}{ccc}
 & V & \\
\swarrow & & \searrow \\
\operatorname{Ker} d_0 & & (\operatorname{Ker} d_0)^{\perp} = U \\
\swarrow \quad \searrow & & \\
d_0 U \qquad W. & &
\end{array}$$

Sullivan pairing.

$(\wedge V, d) \qquad X.$

$m_X : (\wedge V, d) \longrightarrow A_{PL}(X)$

$m_k : (\wedge(e), 0) \rightarrow A_{PL}(S^k)$ or $(\wedge(e, e'), de' = e^2) \rightarrow A_{PL}(S^k)$

$\alpha \in \pi_k(X)$ is represented by $a : (S^k, *) \rightarrow (X, *)$

Let $Q(a) : V^k \rightarrow \Bbbk \cdot e$

Define the pairing $\langle - ; - \rangle : V \times \pi_*(X) \rightarrow \Bbbk$.

$$\langle v ; \alpha \rangle \cdot e = \begin{cases} Q(a) v & v \in V^k \\ 0 & |v| \neq |\alpha| \end{cases}$$

induces a natural linear map : $\nu_X : V \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \Bbbk) \quad v \mapsto \langle v ; - \rangle$

$$\begin{array}{ccc} H^+(\wedge V_X) & \xrightarrow[\cong]{H(m_X)} & H^*(X ; \Bbbk) \\ \zeta \downarrow & & \downarrow hur_X^* \\ V_X & \xrightarrow{\nu_X} & \mathrm{Hom}(\pi_*(X), \Bbbk) \end{array}$$

Cell attachment. $X \cup_a D^{n+1}$. commutative model $(\wedge V \oplus \Bbbk u, d_\alpha)$ by :

$\deg u = n+1$

$\wedge V$ is a subalgebra and $u \cdot \wedge^+ V = 0 = u^2$

$d_\alpha u = 0$, $d_\alpha v = dv + \langle v ; \alpha \rangle u$, $v \in V$.

Differential $d = d_0 + d_1 + d_2 + \cdots$ ($d = d_1 + d_2 + \cdots$ minimal)

$d_1 : V \to \Lambda^2 V.$

$\gamma_0 \in \pi_k(X)$, $\gamma_1 \in \pi_n(X)$ $[\gamma_0, \gamma_1]_W \in \pi_{n+k-1}(X)$

$[c_0, c_1]_W : S^{k+n-1} \to S^k \vee S^n \xrightarrow{(c_0, c_1)} X.$

Define a trilinear map.

$$\langle -; -, - \rangle : \Lambda^2 V \times \pi_*(X) \times \pi_*(X) \longrightarrow \mathbb{k}. \quad \text{by}$$

$$\langle v \wedge w ; \gamma_0, \gamma_1 \rangle = \langle v; \gamma_1 \rangle \langle w ; \gamma_0 \rangle + (-1)^{|w|\cdot|\gamma_0|} \langle v; \gamma_0 \rangle \cdot \langle w ; \gamma_1 \rangle.$$

Prop. $\langle d_1 v; \gamma_0, \gamma_1 \rangle = (-1)^{k+n-1} \langle v; [\gamma_0, \gamma_1]_W \rangle$

Fibration. (Serre).

$$\begin{array}{ccc} F & \xrightarrow{j} & X \\ \downarrow & & \downarrow p \\ y_0 & \longrightarrow & Y \end{array} \quad \text{to} \quad \begin{array}{ccc} A_{PL}(F) & \xleftarrow{A_{PL}(j)} & A_{PL}(X) \\ \uparrow & & \uparrow A_{PL}(p) \\ \mathbb{k} & \xleftarrow{\varepsilon.} & A_{PL}(Y) \end{array}$$

$A_{PL}(p) : A_{PL}(Y) \longrightarrow A_{PL}(X)$ to $m : (A_{PL}(Y) \otimes \Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$

$(\Lambda V, \bar{d}) = \mathbb{k} \otimes_{A_{PL}(Y)} (A_{PL}(Y) \otimes \Lambda V, d)$ fibre of the model at y_0.

$$\begin{array}{ccccc} A_{PL}(Y) & \xrightarrow{A_{PL}(p)} & A_{PL}(X) & \xrightarrow{A_{PL}(j)} & A_{PL}(F) \\ m_Y \uparrow \simeq & & m \uparrow \simeq & & \bar{m} \uparrow \simeq \\ (\Lambda V_Y, d) & \xrightarrow{i} & (\Lambda V_Y \otimes \Lambda V, d) & \xrightarrow{\varepsilon \cdot id.} & (\Lambda V, \bar{d}) \end{array}$$

all are Sullivan models.

• Spatial realization 1·1.

1:1 : Commutative cochain algebras $\rightsquigarrow$ CW complexes.

commutative cochain algebras $\overset{\langle\cdot\rangle}{\rightsquigarrow}$ simplicial sets $\overset{|\cdot|}{\rightsquigarrow}$ CW complexes

(contravariant) Sullivan's simplicial realization functor, adjoint of $A_{PL}(-)$

Milnor's realization functor.

Sullivan realization : the contravariant functor $(A,d) \mapsto \langle A,d \rangle$ from commutative cochain algebras to simplicial sets, give by :

(1) The n-simplices of $\langle A,d\rangle$ are the dga morphisms $\sigma : (A,d) \to (A_{PL})_n$.

(2). face and degeneracy operators are given by $\partial_i \sigma = \partial_i \circ \sigma$ and $s_j \sigma = s_j \circ \sigma$.

(3) If $\varphi : (A,d) \to (B,d)$ is a morphism of commutative cochain algebras, then $\langle \varphi \rangle : \langle A,d\rangle \leftarrow \langle B,d\rangle$ is the simplicial morphism given by

$$\langle \varphi \rangle (\sigma) = \sigma \circ \varphi \qquad \sigma \in \langle B,d\rangle_n.$$

• Lie model.

graded Lie algebra, L, graded vector space $L = \{L_i\}_{i \in \mathbb{Z}}$, $[\ ,\ ] : L \otimes L \to L$

$$+ \quad [x,y] = -(-1)^{\deg x \deg y}[y,x]$$

$$(2) \quad [x,[y,z]] = [[x,y],z] + (-1)^{|x|\cdot|y|}[y,[x,z]]$$

$$(d[x,y] = [dx,y] + (-1)^{|x|}[x,dy])$$

Universal enveloping algebras.

graded Lie algebra $L \longrightarrow$ tensor algebra $TL = \wedge L$.

$\longrightarrow$ universal enveloping algebra of L : $UL = TL/I$

I : generated by the elements of form $x \otimes y - (-1)^{|x|\cdot|y|} y \otimes x - [x,y]$
$x, y \in L$.

$$\begin{array}{ccc} E & \xrightarrow{\iota} & UE \\ \varphi \downarrow & & \downarrow U\varphi \\ L & \xrightarrow{\iota} & UL \end{array}$$

admissible U-monomials : 1, and $u_M = (\iota x_{\alpha_1}) \cdots (\iota x_{\alpha_k}) \in UL$.
$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$.
α_i occurs with multiplicity one if odd.

Thm. 1. admissible U-monomials are a basis of UL.

2. a natural linear isomorphism of graded vector spaces,

$$\gamma : \wedge L \xrightarrow{\cong} UL$$

$$\gamma(x_1 \wedge \cdots \wedge x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}.$$

Free graded Lie algebras $\mathbb{L}_V$.

TV. $[v,w] \triangleq v\otimes w - (-1)^{|v|\cdot|w|} w\otimes v$.

$\mathbb{L}_V$ is a sub Lie algebra generated by V.

inclusion $\mathbb{L}_V \to TV$ extend to an algebra morphism $U\mathbb{L}_V \to TV$.

inclusion $V \hookrightarrow \mathbb{L}_V \hookrightarrow U\mathbb{L}_V$ extend to a morphism $TV \to U\mathbb{L}_V$

$\Rightarrow$ $U\mathbb{L}_V = TV$.

The homotopy Lie algebra of a topological space

$$[\alpha,\beta] = (-1)^{|\alpha|+1} \partial_*([\partial_*^{-1}\alpha, \partial_*^{-1}\beta]_W) \qquad \alpha,\beta \in \pi_*(\Omega X)$$

$\pi_*(\Omega X)\otimes \Bbbk$ is a graded Lie algebra., denoted by L_X.

The homotopy Lie algebra of a minimal Sullivan algebra.

$(\Lambda V, d = d_1 + d_2 + \cdots)$, consider $(\Lambda V, d_1)$

$$sL = \mathrm{Hom}(V, \Bbbk) \qquad (sL)_k = L_{k-1}$$

Define a pairing $\langle -;- \rangle : V \times sL \to \Bbbk$ $\quad \langle v; sx\rangle = (-1)^{|v|} sx(v)$

extend to $(k+1)$-linear maps.

$$\Lambda^k V \times sL \times \cdots \times sL \to \Bbbk$$

$$\langle v_1 \wedge \cdots \wedge v_k ; sx_k, \cdots, sx_1\rangle = \sum_{\sigma\in S_k} \varepsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle \cdots \langle v_{\sigma(k)}; sx_k\rangle$$

consider a pair of dual basis for V and for L, (v_i) for V, (x_j) for L,

s.t. $\langle v_i ; sx_j\rangle = \delta_{ij}$

$[\ ,\ ] : L\times L \to L$ is uniquely determined by the formula

$$\langle v; s[x,y]\rangle = (-1)^{|y|+1}\langle d_1 v; sx, sy\rangle, \quad x,y\in L,\ v\in V.$$

$(L, [-,-])$ is called the homotopy Lie algebra of the Sullivan algebra $(\Lambda V, d)$

Thm. linear map $\sigma : L_X \longrightarrow L$ defined by $\theta(s\alpha) = s\sigma\alpha$, $\alpha \in L_X$.

is an isomorphism of graded Lie algebras.

$$\left(\theta : \pi_*(X) \otimes \mathbb{k} \longrightarrow \mathrm{Hom}(V, \mathbb{k}) \right).$$

- functors C_* and $\mathcal{L}$

$$\left\{ \begin{array}{c} \text{one-connected cocommutative} \\ \text{chain coalgebras} \end{array} \right\} \underset{C_*}{\overset{\mathcal{L}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{connected chain} \\ \text{Lie algebras} \end{array} \right\}$$

$$\mathcal{L}(C_*(L, d_L)) \xrightarrow{\simeq} (L, d_L) \qquad C_*(\mathcal{L}(C,d)) \xleftarrow{\simeq} (C, d)$$

(one-connected : $C = \mathbb{k} \oplus \{C_i\}_{i \geq 2}$, connected chain Lie algebra : $L = \{L_i\}_{i \geq 1}$)

(comultiplication $\Delta : C \to C \otimes C$, augmentation $\varepsilon : C \to \mathbb{k}$
s.t. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ and $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id_C$)

~~$\bar{C} = \ker \varepsilon$ s.t. $C = \mathbb{k} \oplus \bar{C}$~~

(cocommutative if $\tau\Delta = \Delta$, $\tau : C \otimes C \to C \otimes C$, $a \otimes b \mapsto (-1)^{|a|\cdot|b|} b \otimes a$)

for, ΛV , $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon : \Lambda^+ V \to 0$, $1 \to 1$

consider (L, d_L) , define d_0, d_1,

$$d_0(sx_1 \wedge \cdots \wedge sx_k) = -\sum_{i=1}^{k} (-1)^{n_i} sx_1 \wedge \cdots \wedge s d_L x_i \wedge \cdots \wedge sx_k$$

$$d_1(sx_1 \wedge \cdots \wedge sx_k) = \sum_{1 \leq i < j \leq k} (-1)^{|x_i|+1} (-1)^{n_{ij}} s[x_i, x_j] \wedge sx_1 \cdots \hat{sx_i} \cdots \hat{sx_j} \cdots \wedge sx_k.$$

$$\left(n_i = \sum_{j<i} \deg sx_j \qquad sx_1 \wedge \cdots \wedge sx_k = (-1)^{n_{ij}} sx_i \wedge sx_j \wedge sx_1 \cdots \hat{sx_i} \cdots \hat{sx_j} \cdots \wedge sx_k \right)$$

The Cartan-Eilenberg-Chevallery construction on a dgl (L, d_L) is the differential graded coalgebra

$$C_*(L, d_L) = (\Lambda sL, d_0 + d_1)$$

Quillen functor $\mathcal{L}$

$(C,d) = (\bar{C}, d) \oplus \mathbb{k}$ co-augmented dgc, cocommutative.

by cobar construction, $\Omega C = T s^{-1}\bar{C}$, $d = d_0 + d_1$

$d_0 : s^{-1}\bar{C} \to s^{-1}\bar{C}$, $d_1 : s^{-1}\bar{C} \to s^{-1}\bar{C} \otimes s^{-1}\bar{C}$

$\mathcal{L}(C,d) = (\mathbb{L}_{s^{-1}\bar{C}}, d)$ ~~dgl~~.

Def. free model of a connected chain Lie algebra (L,d) is a dgl quasi-isomorphism of the form.

$$m : (\mathbb{L}_V, d) \xrightarrow{\simeq} (L,d) \quad \text{with } V = \{V_i\}_{i \geq 1}$$

$$\varphi : \mathcal{L}C_*(L,d) \xrightarrow{\simeq} (L,d)$$

Prop. $\varphi : (\mathbb{L}_W, d) \to (\mathbb{L}_V, d)$ ~~connected~~ morphism of free connected chain Lie algebras

then

$$\varphi : \simeq \iff \varphi_0 : \simeq \qquad (\varphi_0 = Q(\varphi))$$

Def. minimal : $(\mathbb{L}_V, d)$ $V = \{V_i\}_{i \geq 1}$, if $d = d_0 + d_1 + \cdots$, $d_0 = 0$

$m : (\mathbb{L}_V, d) \xrightarrow{\simeq} (L,d)$ is called a minimal free Lie model.

Thm. For any connected chain algebra (L,d), admits a minimal free Lie model

$$m : (\mathbb{L}_V, d) \xrightarrow{\simeq} (L,d)$$

and $(\mathbb{L}_V, d)$ is unique up to isomorphism.

$C^*(L, d_L)$ and $\mathcal{L}_{(A,d)}$

$C^*(L, d_L) = \mathrm{Hom}(C_*(L, d_L), \mathbb{k})$ (Commutative , dga)

$(f\cdot g)(c) = (f\otimes g)(\Delta c)$ $(df)(c) = -(-1)^{|f|} f(dc)$ $f, g \in C^*(L, d_L)$
$c \in C_*(L, d_L)$

If (L, d) is a connected chain Lie algebra, and each L_i is finite dimensional

consider $(sL)^\# = \mathrm{Hom}(sL, \mathbb{k}) \hookrightarrow C^*(L)$, extend to $\varphi: \Lambda(sL)^\# \xrightarrow{\cong} C^*(L)$

which exhibits $C^*(L)$ as a Sullivan algebra.

Suppose $A = \mathbb{k} \oplus A^{\geq 2}$ is a commutative cochain algebra, A^i is finite dimensional,

$(C, d_C) = \mathrm{Hom}(A, \mathbb{k})$, $\mathcal{L}_{(A,d)} \triangleq \mathcal{L}(C, d_C)$

we have results : $C^*(\mathcal{L}_{(A,d)}) \xrightarrow{\simeq} (A, d)$

$C^*(\mathcal{L}_{(A,d)})$ as a functorial Sullivan model of (A, d)

Example. Minimal Lie models of minimal Sullivan algebras.

$(\Lambda W, d)$ is a minimal Sullivan algebra, and that $W = \{W^i\}_{i\geq 2}$ is a graded vector space of finite type.

let $(\mathbb{L}_V, \partial)$ be a minimal Lie model of $\mathcal{L}_{(\Lambda W, d)}$, then $(\Lambda W, d)$ is a minimal Sullivan model for $C^*(\mathbb{L}_V, \partial)$

Lie model for topological spaces and CW-complexes

Def. Lie model for X is a connected chain Lie algebra (L, d_L) of finite type

s.t. $m: C^*(L, d_L) \xrightarrow{\simeq} A_{PL}(X)$

free Lie model for X: Lie model + "free", $L = \mathbb{L}V$.

Lie representative for a continue map $f: X \to Y$ is a dgl morphism

$$\varphi: (L, d_L) \to (E, d_E), \text{ s.t. } m C^*(\varphi) \sim A_{PL}(f)\, n.$$

Example. S^k

$$\mathbb{L}(v) = \begin{cases} \Bbbk v & |v| = 2n \\ \Bbbk v \oplus \Bbbk [v,v] & |v| = 2n+1 \end{cases}$$

$$C^*(\mathbb{L}(v)) = \begin{cases} (\Lambda(e), 0) & |e| = 2n+1 \\ (\Lambda(e, e'), de' = e^2) & |e| = 2n+2 \end{cases}$$

Conclusion:

- Every space X has a minimal free Lie model, unique up to isomorphism, and every continuous map has a Lie representative.
- Every connected chain Lie algebra, (L, d_L) of finite type, and [illegible] defined over $\mathbb{Q}$, is the Lie model of a [illegible] simple connected CW complex, unique up to rational homotopy equivalence.
- If (L, d_L) is a Lie model for X, then there a natural isomorphism

$$H(L) \xrightarrow{\simeq} \pi_*(\Omega X) \otimes_{\mathbb{Z}} \Bbbk \quad \text{or} \quad sH(L) \xrightarrow{\simeq} \pi_*(X) \otimes_{\mathbb{Z}} \Bbbk$$

If $(L, d_L) = (\mathbb{L}V, d)$ for X, then there a isomorphism.

$$sH(V, dv) \oplus \Bbbk \simeq H_*(X; \Bbbk)$$

with some hypotheses above the diagrame commutes

$$\begin{array}{ccc} sH(\mathbb{L}_V, d) & \xrightarrow[\cong]{\tau_L} & \pi_*(X)\otimes \mathbb{k} \\ {\scriptstyle sH(\eta)}\downarrow & & \downarrow {\scriptstyle hur\, X} \\ sH(V, d_V) & \xrightarrow[\cong]{} & H_+(X; \mathbb{k}) \end{array}$$

Lie models for adjunction spaces. (the existence of free Lie model)

Consider $Y = X \coprod_\alpha e^{n_\alpha+1} = X \cup_f (\coprod_\alpha D^{n_\alpha+1})$ where:

(1) X is simply connected with rational homotopy of finite type.

(2) $f = \{ f_\alpha : (S^{n_\alpha}, *) \to (X, x_0)\}$

(3) the cell $D^{n_\alpha+1}$ are all of dimension ≥ 2, with finitely many in any given dimension.

Suppose $m : C^*(\mathbb{L}_V, d) \to A_{PL}(X)$ is a free Lie-model for X.

we shall constructure a free Lie model for Y.

constructure: given an isomorphism. $\tau_L : sH(\mathbb{L}_V) \xrightarrow{\cong} \pi_*(X)\otimes \mathbb{k}$

the classes $[f_\alpha] \in \pi_{n_\alpha}(X)$ determine classes $s[z_\alpha] = \tau_L^{-1}[f_\alpha] \in sH(\mathbb{L}_V)$, $z_\alpha \in \mathbb{L}_V$. Let W be a graded vector space with basis $\{w_\alpha\}$ and $|w_\alpha| = n_\alpha$

we can extend $\mathbb{L}_V$ to a chain Lie algebra $\mathbb{L}_{V\oplus W} = \mathbb{L}(V\oplus W)$ by defining $dw_\alpha = z_\alpha$

Thm. the chain Lie algebra $(\mathbb{L}_{V\oplus W}, d)$ is a Lie model for Y.

n-skeleton, X_n, $(\mathbb{L}_{V_{\leq n-1}}, d)$ is identified as a Lie model for X_n

$$sH_*(\mathbb{L}_{V_{\leq n-1}}, d) \xrightarrow{\cong} \pi_*(X_n)\otimes \mathbb{Q}$$

Example.

1. a wedge of spheres. $X = \bigvee_\alpha S^{n_\alpha+1} = pt \cup_f \left(\coprod_\alpha D^{n_\alpha+1}\right)$

$(\mathbb{L}_V, 0)$, $V = \{V_i\}_{i\geq 1}$ basis $\{v_\alpha\}$, $|v_\alpha| = n_\alpha$.

2. the free product of Lie models is a Lie model for wedge, $\bigvee_\alpha X_\alpha$

$X = \bigvee_\alpha X_\alpha$ finite type. $(\mathbb{L}_{V(\alpha)}, d_\alpha)$ be a Lie model for X_α

$\coprod_\alpha (\mathbb{L}_{V(\alpha)}, d_\alpha) \cong (\mathbb{L}_{\oplus_\alpha V(\alpha)}, d)$ is a Lie model for $\bigvee_\alpha X_\alpha$.

$$\pi_*(\Omega \bigvee_\alpha X_\alpha) \otimes \mathbb{Q} = H(\coprod_\alpha \mathbb{L}_{V(\alpha)}, d_\alpha) = \coprod_\alpha \pi_*(\Omega X_\alpha)$$

(Free product of $\{L(\alpha)\}_{\alpha\in J}$, $L(\alpha)$ is a graded Lie algebra,

$\coprod_\alpha L(\alpha) \triangleq \mathbb{L}_V / I$ $V = \bigoplus_\alpha L(\alpha)$, $I \subset \mathbb{L}_V$ be the ideal generated by

$i_\alpha[x,y] - [i_\alpha x, i_\alpha y]$, $x, y \in L(\alpha)$, $\alpha \in J$. $i_\alpha : L(\alpha) \to V$.)

3. Let (L_α, d_α) be Lie models for simply connected space X_α, s.t.

$X = \prod_\alpha X_\alpha$ is finite type.

$\bigoplus_\alpha (L_\alpha, d_\alpha)$ is a Lie model for X.

4. $f : X \to Y$. Lie representative for f : $\varphi : (L, d_L) \to (K, d_K)$

let $0 \to (I, d_I) \to (L, d_L) \to (K, d_K) \to 0$

be a short exact sequence of differential graded Lie algebras (connected finite type).

(I, d_I) is a Lie model for the homotopy fibre of f.