Motivations to Develop Spectral Sequences

Tongtong Liang

Abstract

This is a short note on spectral sequences focusing on the motivation to develop spectral sequences. The main reference is [Cho06].

Given a chain complex $\{C_\bullet\}$ then

$$\cdots \xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots$$

and assume that C_d is finitely generated. Our goal is to compute the homology groups. Sometimes, a chain complex is very complicated, it is very hard to compute homology so we may expect to **divide and conquer**. More specifically, for each C_d, we give a filtration

$$0 = C_{d,0} \subseteq C_{d,1} \subseteq \cdots \subseteq C_{d,n} = C_d$$

such that $\partial C_{d,p} \subseteq C_{d-1,p}$. The diagram is

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & C_{d+1,n} & \xrightarrow{\partial} & C_{d,n} & \xrightarrow{\partial} & C_{d-1,n} & \xrightarrow{\partial} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C_{d+1,n-1} & \longrightarrow & C_{d,n-1} & \longrightarrow & C_{d-1,n-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C_{d+1,1} & \longrightarrow & C_{d,1} & \longrightarrow & C_{d-1,1} & \longrightarrow & \cdots
\end{array}$$

Let $E^0_{d,p} := C_{d,p}/C_{d,p-1}$, then $\oplus_{p=1}^n E^0_{d,p}$ is a good approximation of C_d and naturally, we have

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & E^0_{d+1,n} & \xrightarrow{\partial^0} & E^0_{d,n} & \xrightarrow{\partial^0} & E^0_{d-1,n} & \longrightarrow & \cdots \\
 & & & & & & & & \\
\cdots & \longrightarrow & E^0_{d+1,n-1} & \xrightarrow{\partial^0} & E^0_{d,n-1} & \xrightarrow{\partial^0} & E^0_{d-1,n-1} & \longrightarrow & \cdots \\
 & & & & & & & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & & & & & & & \\
\cdots & \longrightarrow & E^0_{d+1,1} & \xrightarrow{\partial^0} & E^0_{d,1} & \xrightarrow{\partial^0} & E^0_{d-1,1} & \longrightarrow & \cdots
\end{array}$$

Now let

$$E^1_{d,p} := H_d(E^0_{\bullet,p}) = \frac{\ker \partial^0 : E^0_{d,p} \to E^0_{d-1,p}}{\operatorname{im} \partial^0 : E^0_{d+1,p} \to E^0_{d,p}}$$

Question:

$$\bigoplus_{p=1}^n E^1_{d,p} \cong H_d(C_\bullet)?$$

Even though for each d, $\oplus_{p=1}^n E^0_{d,p} \cong C_d$, the isomorphisms may fail, because the isomorphisms are not at the level of chain complexes. Let's figure out the gap.

Consider a special case where $n = 2$, then we have

$$0 = Z_{d,0} \subseteq Z_{d,1} \subseteq Z_{d,2} = Z_d$$

where $Z_{d,i} = Z_d \cap C_{d,i}$, and

$$0 = B_{d,0} \subseteq B_{d,1} \subseteq B_{d,2} = B_d$$

where $B_{d,i} = B_d \cap C_{d,i}$, then

$$H_d(C_\bullet) = \frac{Z_d}{B_d} \cong \frac{Z_d + C_{d,1}}{B_d + C_{d,1}} \oplus \frac{Z_d \cap C_{d,1}}{B_d \cap C_{d,1}} = \frac{Z_d + C_{d,1}}{B_d + C_{d,1}} \oplus \frac{Z_{d,1}}{B_{d,1}}$$

Now the question may be turned into

$$E^1_{d,1} \cong \frac{Z_{d,1}}{B_{d,1}}?$$

and

$$E^1_{d,2} \cong \frac{Z_d + C_{d,1}}{B_d + C_{d,1}}?$$

However, both of them still may fail! Let's see how them fail.

First, compare

$$E^1_{d,1} = \frac{\ker \partial^0 : E^0_{d,1} \to E^0_{d-1,1}}{\operatorname{im} \partial^0 : E^0_{d+1,1} \to E^0_{d,1}} = \frac{\ker \partial^0 : C_{d,1} \to C_{d-1,1}}{\operatorname{im} \partial^0 : C_{d+1,1} \to C_{d,1}}$$

and

$$\frac{Z_{d,1}}{B_{d,1}} = \frac{(\ker \partial : C_d \to C_{d-1}) \cap C_{d,1}}{(\operatorname{im} \partial : C_{d+1} \to C_d) \cap C_{d,1}}$$

note that

$$(\ker \partial : C_d \to C_{d-1}) \cap C_{d,1} = \ker \partial^0 : C_{d,1} \to C_{d-1,1}$$

and

$$\operatorname{im} \partial^0 : C_{d+1,1} \to C_{d,1} \subseteq (\operatorname{im} \partial : C_{d+1} \to C_d) \cap C_{d,1}$$

however, it is possible that

$$\operatorname{im} \partial^0 : C_{d+1,1} \to C_{d,1} \neq (\operatorname{im} \partial : C_{d+1} \to C_d) \cap C_{d,1}$$

because there may exist an element $x \in C_{d,2} \setminus C_{d,1}$ such that $\partial(x) \in C_{d,1}$

$$\begin{array}{ccccc} x \in C_{d+1,2} \setminus C_{d+1,1} & \hookrightarrow & C_{d+1,2} & \xrightarrow{\partial} & C_{d,2} \\ & \searrow & \uparrow & & \uparrow \\ & & C_{d+1,1} & \xrightarrow[\partial]{} & C_{d,1} \end{array}$$

We say x "go downstairs". Therefore, $Z_{d,1}/B_{d,1}$ **is a quotient of** $E^1_{d,1}$**.**

Then, we compare

$$E^1_{d,2} = \frac{\ker \partial^0 : C_{d,2}/C_{d,1} \to C_{d-1,2}/C_{d-1,1}}{\operatorname{im} \partial^0 : C_{d+1,2}/C_{d+1,1} \to C_{d,2}/C_{d,1}}$$

and

$$\frac{Z_d + C_{d,1}}{B_d + C_{d,1}}$$

note that $[x] = [y]$ in $E^1_{d,2}$ if and only if $x - y \in C_{d,1} + B_{d,2}$. However, if $[x] \in \ker \partial^0 : C_{d,2}/C_{d,1} \to C_{d-1,2}/C_{d-1,1}$, x may be in $\partial^{-1}(C_{d-1,1}) \setminus \ker \partial$ i.e.

$$\partial^{-1}(C_{d-1,1}) + C_{d,1} \neq Z_{d,2} + C_{d,1}$$

is possible. Such x "goes downstairs"! Hence

$$\frac{Z_d + C_{d,1}}{B_d + C_{d,1}} \subseteq E^1_{d,2}$$

In summary, the key point the elements that "go downstairs". More specifically, the elements in $C_{d+1,2}$ that "go downstairs" effect the possible gap between $E^1_{d,1}$ and $\frac{Z_{d,1}}{B_{d,1}}$; the elements in $C_{d,2}$ that "go downstairs" effect the possible gap between $E^1_{d,2}$ and $\frac{Z_d+C_{d,1}}{B_d+C_{d,1}}$.

Fortunately, the gaps can be fixed by considering the "homology groups of homology groups", due to the following observation: the elements which are not cycles and go downstairs are hidden in $\partial^{-1}(C_{d-1,1})$ while the cycles are in $Z_{d,2}$, hence

$$\frac{\partial^{-1}(C_{d-1,1})}{Z_{d,2}}$$

is exactly the gap and now we need to fix the gap in a proper way. Note that for each $[x] \in \ker \partial^0 : C_{d,2}/C_{d,1} \to C_{d-1,2}/C_{d-1,1}$, $x \in \partial^{-1}(C_{d-1,1})$, hence there is a natural map

$$\partial^1 : E^1_{d,2} \to E^1_{d-1,1}$$

(we may check that such ∂^1 is well-defined: if $[x] = [y]$ in $E^1_{d,2}$, then $x - y \in B_{q,2} + C_{q,1}$, then $\partial(x - y) \in \partial(C_{d,1}) \subseteq B_{d,1}$, then $[\partial(x)] = [\partial(y)]$ in $E^1_{d-1,1}$.) Here we give to claim to figure out how these maps fix the gaps:

Claim 1: $E^1_{d,1}/\operatorname{im}\partial^1 \cong \frac{Z_{d,1}}{B_{d,1}}$. We have shown that $\frac{Z_{d,1}}{B_{d,1}}$ is a quotient of $E^1_{d,1}$. If $\alpha \in \operatorname{im}\partial^1$, then there exits $\alpha' \in \partial^{-1}(C_{d,1})$ such that $\partial(\alpha') = \alpha$, hence $\operatorname{im}\partial^1$ fixes the gap between $\operatorname{im}(\partial : C_{d+1,1} \to C_{d,1})$ and $\operatorname{im}(C_{d+1,2} \to C_{d,2}) \cap C_{d,1}$ and

$$E^1_{d,1}/\operatorname{im}\partial^1 \cong \frac{Z_{d,1}}{B_{d,1}}$$

Claim 2: $\ker \partial^1 \cong \frac{Z_d+C_{d,1}}{B_d+C_{d,1}}$.

Note that we have shown that

$$\frac{Z_d + C_{d,1}}{B_d + C_{d,1}} \subseteq E^1_{d+1,2}$$

and clearly,

$$\frac{Z_d + C_{d,1}}{B_d + C_{d,1}} \subseteq \ker \partial^1$$

conversely, if $[x] \in \ker \partial^1$, then $\partial(x) \in \operatorname{im}(C_{d+1,1} \to C_{d,1})$, then we can find $x' \in C_{d+1,1}$ such that $\partial(x - x') = 0$ i.e. $x - x' \in Z_{d+1,1} \subseteq C_{d+1,1}$ and $[x] = [x']$ in $E^1_{d+1,2}$ and $[x'] \in \frac{Z_{d+1,2}+C_{d+1,1}}{B_{d+1,2}+C_{d+1,1}}$.

Hence we may consider the diagram:

$$
\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & \\
& \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & & \\
\cdots & & E^1_{d+1,2} & & E^1_{d,2} & & E^1_{d-1,2} & & \cdots & & \\
& \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & & \\
\cdots & & E^1_{d+1,1} & & E^1_{d,1} & & E^1_{d-1,1} & & \cdots & & \\
& \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & \overset{\partial^1}{\searrow} & & & \\
0 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{1}
$$

The diagram consists of chain complex clearly if we consider the slant downwards arrows. Then we define

$$E^2_{d,p} : H_d(E^1_{d+\bullet,p+\bullet}) = \frac{\ker \partial^1 : E^1_{d,p} \to E^1_{d-1,p-1}}{\operatorname{im} \partial^1 : E^1_{d+1,p+1} \to E^1_{d,p}}$$

and according to claim 1 and claim 2, we have

$$\bigoplus_{p=1}^{2} E^2_{d,p} \cong H_d(C_\bullet)$$

The case that $n = 2$ completes. For general cases, naturally, we also have

$$\partial^2 : E^2_{d+1,p+2} \to E^2_{d,p}$$

and we have a branch of chain complexes as diagram 1, then we can define E^3 as homology groups of corresponding E^2-chain complex. Procceed the procedure, we have $E^3, E^4, E^5, \ldots, E^n$. Inductively, E^n is the correct answer of $H_\bullet(C_\bullet)$.

Informal explanation: No matter what n is, the key point is the elements that go downstairs. When $n = 2$, an element at most go down two steps and E^2 catches all the information of the elements that go down no more than two steps. For $n > 2$, an element may go down $p > 2$ steps, then we may use E^p to take it into account, and so on, E^n page gives the correct homology groups for a filtration of length n. In other words (still informally speaking), E^p is a p-order approximation of the desired homology groups while the error between the homology groups of an n-filtration of a chain complex and the original homology group is at most of order n.

References

[Cho06] Timothy Y. Chow, *You could have invented spectral sequences*, Notices Amer. Math. Soc. **53** (2006), no. 1, 15–19.