K – Theory (equivariant)

— References

- Hatcher A. Vector bundles and K-theory
- Segal G. Equivariant K-theory

— Vector bundles

E, B — topological spaces together with

$p : E \longrightarrow B$ satisfying

(1) $\forall\, b \in B$, $p^{-1}(b)$ is a vector space.

(2) $\forall\, b \in B$, $\exists\, U$, an open nbhd of b, s.t. diagram commutes.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\text{homeo}", "\simeq"'] \arrow[dr] & & U \times \mathbb{C}^n \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

— Examples

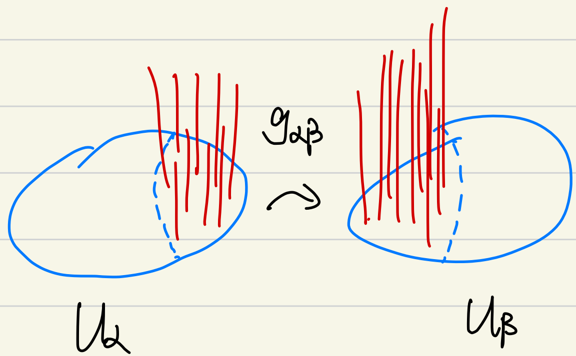
- The trivial bundle $B \times \mathbb{C}^n$.
- Tangent bundle of a differential manifold.
- Mobius band $M \to S^1$, $[0,1] \times \mathbb{R} / (0,t) \sim (1,-t)$.

Its orthogonal bundle $M^\perp \subseteq \mathbb{R}^3$ consisting of all "orthogonal lines" is isomorphic to M.

For bundles E_1, E_2 over B, we can take direct sum

$E_1 \oplus E_2$ and tensor product $E_1 \otimes E_2$.

Γ $E_1 \to B$, for U_α and U_β, there is a map $g^1_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}^n)$ which tell us how to patch such trivial pieces together; a way to construct bundles.



For $E_2 \to B$, $g^2_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}^n)$. then for $E_1 \oplus E_2 \to B$

$g_{\alpha\beta}$ should be $g^1_{\alpha\beta} \oplus g^2_{\alpha\beta}$ $\left[\begin{array}{c|c} g^1_{\alpha\beta} & \\ \hline & g^2_{\alpha\beta} \end{array}\right]$

Similarly, $E_1 \otimes E_2$ could construct from $g_{\alpha\beta} = g^1_{\alpha\beta} \otimes g^2_{\alpha\beta}$:

$U_\alpha \cap U_\beta \to GL(\mathbb{C}^n)$

$\lrcorner$

- Remark

- nontrivial $\oplus$ nontrivial may be trivial. ($M \oplus M$)
- nontrivial $\oplus$ trivial may be trivial. $\left(TS^n \oplus NS^n \right)$

Assume B is compact Hausdorff from now on.

— Fact 1 : $\forall E \to B$, $\exists E' \to B$ such that $E \oplus E'$ is trivial

Suppose $f: A \to B$, $E \to B$, then there is $E' \to A$ fitting into the diagram with $\check{f}: p^{-1}(a) \to p^{-1}(f(a))$ is isomorphism for each $a \in A$.

$$\begin{tikzcd} E' \arrow[r, "\check{f}"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Such E' is unique up to isomorphism, E' is just the fiber product of

$$\begin{tikzcd} & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

.

Let $\mathrm{Vect}_{\mathbb{C}}(B)$ denote all bundles over B (up to iso)

then $f: A \to B$ induces $f^*: \mathrm{Vect}_{\mathbb{C}}(B) \longrightarrow \mathrm{Vect}_{\mathbb{C}}(A)$ with

- $(fg)^*(E) \cong g^* f^*(E)$,
- $\mathbb{1}^*(E) \cong E$,
- $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$, and
- $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$.

Moreover if $f_0 \sim f_1 : A \to B$, then $f_0^* = f_1^* : \mathrm{Vect}_{\mathbb{C}}(B) \to$

$\mathrm{Vect}_{\mathbb{C}}(A)$. This leads us to the definition of K-groups.

— K-Theory.

$\varepsilon^n \to X$, n-dimensional trivial bundle over X.

$E_1 \sim E_2$ if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$ for some m and n.

⌜ Under $\sim$ relation, $\mathrm{Vect}_{\mathbb{C}}(X)$ forms a group, $\tilde{K}(X)$. ε^0 is the zero element. ⌟

$E_1 \cong_s E_2$ if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^n$ for some n. To make $\mathrm{Vect}_{\mathbb{C}}(X)$

become a group under $\cong_s$, consider all former differences $E-E'$

we say $E_1-E_1' = E_2-E_2'$ iff $E_1\oplus E_2' \cong_s E_2\oplus E_1'$. Then we call this

group $K(X)$. $(E_1-E_1')+(E_2-E_2') = (E_1\oplus E_2)-(E_1'\oplus E_2')$.

- The ring structure on $K(X)$ is also obvious,

$$(E_1-E_1')\otimes(E_2-E_2') = E_1\otimes E_2 - E_1\otimes E_2' - E_1'\otimes E_2 + E_1'\otimes E_2'.$$

Hence $K(X)$ is a commutative ring. Now if $f: X\to Y$,

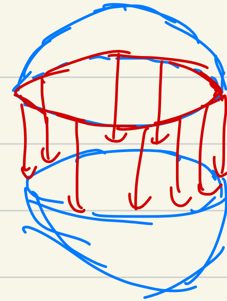
$f^*: K(Y) \to K(X)$ satisfying those properties. Hence

$K(\cdot)$ is a functor : $Htp \longrightarrow Rings$

- Remark: $x_0 \to X$, $K(X) \to K(x_0) \cong \mathbb{Z}$, the kernel can be identified with $\tilde{K}(X)$ and $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$, (cohomology theory and reduced cohomology theory.)

- Examples. D^n can only admit trivial bundles for $D^n \simeq \{pt\}$. Hence for given $f: S^{k-1} \to GL(\mathbb{C}^n)$, we can construct $E_f \to S^n$. In fact,

$[S^{k-1}, GL(\mathbb{C}^n)] \xleftrightarrow{\text{bij}} Vect^n_{\mathbb{C}}(S^k)$.



Let $k=1$, S^1 can only admit trivial bundle,

therefore $K(S^1) = \mathbb{Z}$.

As for $k=2$, $S^2 = \mathbb{C}P^1$, let $H \to \mathbb{C}P^1$ be the canonical line bundle, i.e. $H = \{(v, l) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid v \in l\}$.

$\mathbb{C}P^1 : [z_0, z_1] \to [z_0/z_1, 1] = \mathbb{C} \cup \{\infty\} \simeq S^2$, under this situation, $D^2_0 = [z, 1]$, $|z| \leq 1$ and D^2_∞ can be written as

$[1, z_1/z_0] = [1, z^{-1}]$ with $|z| \leq 1$. $\lambda(z,1) = \lambda z(1, z^{-1})$, hence $f: S^1 \to GL(\mathbb{C})$ is just $f(z) = z$. Now, $H^2+1 = (H \otimes H) \oplus 1$ has clutching function: $\begin{bmatrix} z^2 & \\ & 1 \end{bmatrix}$ and $2H = H \oplus H$ has clutching function $\begin{bmatrix} z & \\ & z \end{bmatrix}$. Since $GL(\mathbb{C}^2)$ is path-connected, there is a path α_t connecting the ~~出~~ identity matrix to the matrix interchange two factors of $\mathbb{C} \times \mathbb{C}$. Therefore $(z \oplus 1)\alpha_t(1 \oplus z)\alpha_t$ connecting $z \oplus z$ and $z^2 \oplus 1$. Hence there is a relation $H^2+1 = 2H$, or $(H-1)^2 = 0$.

in $K(S^2)$, hence we get a map:

$$\mathbb{Z}[H]/(H-1)^2 \longrightarrow K(S^2) \qquad (*)$$

and in fact, this is an isomorphism. As usual,

$$\begin{tikzcd} & X \times Y \arrow[dl, "pr_X"'] \arrow[dr, "pr_Y"] & \\ X & & Y \end{tikzcd} \xrightarrow{K(\cdot)} \begin{tikzcd} & K(X \times Y) & \\ K(X) \arrow[ur, "pr_X^*"] & & K(Y) \arrow[ul, "pr_Y^*"'] \\ & K(X) \otimes K(Y) \arrow[ul] \arrow[ur] \arrow[uu, dashed] & \end{tikzcd}$$

$$a \otimes b \longmapsto pr_X^*(a) \cdot pr_Y^*(b)$$

- **Fundamental theorem:**

$$K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X) \otimes K(S^2) \longrightarrow K(S^2 \times X)$$

is an isomorphism. Hence $K(S^2) = \mathbb{Z}[H]/(H-1)^2$.

— Cohomology Theory.

$$A \longrightarrow X \longrightarrow X/A \quad \text{induces} \quad \tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A)$$

which is exact. Then consider the cofibration sequence:

$$
\begin{tikzcd}
A \arrow[r] & X \arrow[r] & X \cup CA \arrow[r] \arrow[d] & (X\cup CA)\cup CX \arrow[r] & {[(X\cup CA)\cup CX]} \cup C(X\cup CA) \arrow[r] & {} \\
A \arrow[r] & X \arrow[r] & X/A \arrow[r] & SA \arrow[r] & SX \arrow[r] & S(X/A) \arrow[r] & \cdots
\end{tikzcd}
$$

therefore there is a long exact sequence :

$$\cdots \to \tilde{K}(S(X/A)) \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$$

Now suppose $X = A \vee B$, then $\tilde{K}(X) \to \tilde{K}(A)$ is surjective

so does $\tilde{K}(SX) \to \tilde{K}(SA)$, hence

$$\tilde{K}(SX) \to \tilde{K}(SA) \xrightarrow{0} \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A) \to 0$$

$$\tilde{K}(X/A) = \tilde{K}(B)$$

splits into $\tilde{K}(A \vee B) = \tilde{K}(A) \oplus \tilde{K}(B)$.

As we mentioned above $\tilde{K}(X)$ could be identified with the kernel of $K(X) \to K(x_0)$, hence $\tilde{K}(X)$ can be regarded as an ideal of $K(X)$ vanishing on $K(x_0)$.

Therefore $\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow K(X \times Y)$ restricting zero on

$K(X\times\{y_0\}\cup\{x_0\}\times Y)=K(X\vee Y)=K(X)\oplus K(Y)$, which yields a map $\tilde{K}(X)\otimes\tilde{K}(Y)\longrightarrow\tilde{K}(X\wedge Y)$. $a\otimes b\mapsto a*b=pr_x^*(a)\,pr_y^*(b)$

$$\begin{array}{ccccccccc}
K(X)\otimes K(Y) & \cong & \tilde{K}(X)\otimes\tilde{K}(Y) & \oplus & \tilde{K}(X) & \oplus & \tilde{K}(Y) & \oplus & \mathbb{Z}\\
\downarrow & & \downarrow & & \| & & \| & & \|\\
K(X\times Y) & \cong & \tilde{K}(X\wedge Y) & \oplus & \tilde{K}(X) & \oplus & \tilde{K}(Y) & \oplus & \mathbb{Z}
\end{array}$$

$$\underbrace{\tilde{K}(X)\oplus\tilde{K}(Y)}_{\tilde{K}(X\vee Y)}$$

— Bott periodicity $\beta:\ \tilde{K}(X)$

$$\begin{array}{c}\tilde{K}(X)\\ \cong\end{array}$$

Now consider $\tilde{K}(S^2)\otimes\tilde{K}(X)\longrightarrow\tilde{K}(S^2\wedge X)=\tilde{K}(S^2X)$

sending $a \in \tilde{K}(X)$ to $(H-1)\otimes a \mapsto (H-1) * a$. Bott periodic theorem just says that β is an isomorphism.

— Remark: $\tilde{K}(S^2)$ viewed as an ideal of $K(S^2) = \mathbb{Z}[H]/(H-1)^2$ is generated by $\overset{\cong \mathbb{Z}}{(H-1)}$, and the ring structure is trivial since $(H-1)^2=0$

Now we going back to the long exact sequence

$$\tilde{K}(S^2A) \to \tilde{K}(S(X/A)) \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$$

writting $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$ and $\tilde{K}^{-n}(X,A) = \tilde{K}(S^n(X/A))$,

it becomes $\tilde{K}^{-2}(A) \to \tilde{K}^{-1}(X,A) \to \tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to \tilde{K}^{0}(X,A) \to$

$$\tilde{K}^0(X,A) \to \tilde{K}^0(X) \to \underset{\tilde{K}^0(A)}{\overset{\Vert}{}} \to \underset{\tilde{K}^1(X,A)}{\overset{\Vert}{}} \to \underset{\tilde{K}^1(X)}{\overset{\Vert}{}} \to \cdots$$

where $\tilde{K}^{2i}(X) = \tilde{K}(X)$ and $\tilde{K}^{2i+1}(X) = \tilde{K}(SX)$ for $i>0$

To summarize :

$$
\begin{tikzcd}
\tilde{K}^0(X,A) \arrow[r] & \tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \arrow[d] \\
\tilde{K}^1(A) \arrow[u] & \tilde{K}^1(X) \arrow[l] & \tilde{K}^1(X,A) \arrow[l]
\end{tikzcd}
$$

- Proposition : Just as in classical cohomology theory,

  $\alpha \in \tilde{K}^i(X)$, $\beta \in \tilde{K}^j(X)$, $\alpha\beta = (-1)^{ij}\beta\alpha$.

- Corollary : $\tilde{K}(S^{2n}) \cong \tilde{K}(S^2) \cong \mathbb{Z}$ generated by $(H-1) * \cdots * (H-1)$

and $\tilde{K}(S^{2n+1}) \cong \tilde{K}(S^1) = 0$

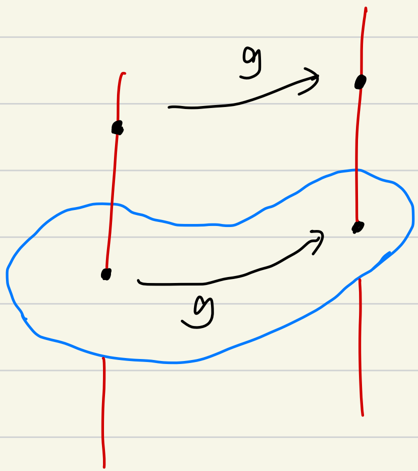
— Equivariant K-Theory.

"K-Theory with group actions"

X a G-space means there is a topo group G acting continuously on X. $1 \cdot x = x$ and $(gg') \cdot x = g \cdot (g' \cdot x)$.

$E \to X$ a G-bundle, if $p: E \to X$ is a vector bundle, where E is a G-space as well and p is a G-map, that is

$\forall g \in G, \quad g \circ p = p \circ g : E \to X \quad (\text{i.e. } g : E_x \to E_{gx})$



An important example is $X = \{pt\}$ has trivial G-action and

$E \to X$ is a G-bundle, then E is a "G-module", i.e. a representation space of G.

— $K_G(\cdot)$

As ordinary K-Theory, consider formal differences $E - E'$ of all G-bundles over a base space X. $E_1 - E_1' = E_2 - E_2'$ iff $\exists$ G-bundle F such that $E_1 \oplus E_2' \oplus F \simeq E_2 \oplus E_1' \oplus F$. Then we denote this group by $K_G(X)$, it is a commutative ring respect to "$\otimes$".

As expected, $K_G(\cdot)$ is a functor from the category of G-spaces and G-maps to the category of Rings, and a few properties still hold.

In particular, If Y is a H-space and X a G-space, and $\alpha: H \to G$, and $\varphi: Y \to X$, $\varphi(h, y) \mapsto (\alpha(h), \varphi(y))$. If E is a G bundle over X, then $\varphi^* E = \{ (y, l) \mid l \in E_{\varphi(y)} \}$ is a H-bundle over Y.

— Examples :

$X = \{pt\}$, $K_G(\{pt\}) = R(G)$ the representation ring of G, generating by all simple G-modules, or equivalently all irreducible representations of G. (not easy to compute)

For a G-space, $X \xrightarrow{\pi} X/G$ induces $K(X/G) \to K_G(X)$, since for each bundle $E \to X/G$, $\pi^*(E)$ admits G-actions

$$g(x,t) = (g\cdot x, t), \quad t \in E_{\pi(x)} = E_{\pi(gx)} = E_{g\pi(x)}.$$

Conversely, if $E \to X$ is a G-bundle and <u>G acts freely</u> on

X, then $E/G \to X/G$ is a bundle (need to verify), and hence

$$K(X/G) \longrightarrow K_G(X) \text{ is an isomorphism.}$$

What if G acts trivially on X?

- $K(X) \longrightarrow K_G(X)$, $E \longmapsto E$ with trivial G action.
- $R(G) \longrightarrow K_G(X)$, inducing by $X \to \{pt\}$.

Together yields $R(G) \otimes K(X) \longrightarrow K_G(X)$ which is an isomorphism.

— Example :

$E \to X$ a bundle, then $E^{\otimes m} \to X^{\times m}$ is naturally a Σ_m-bundle

the diagonal map $\Delta : X \to X^{\times m}$ induces $K_{\Sigma_m}(X^{\times m}) \to K_{\Sigma_m}(X)$. Since

Σ_m acting trivially on $X \cong$ diagonal image, it follows that

$P_m : K(X) \xrightarrow{\wp_m} K_{\Sigma_m}(X^{\times m}) \xrightarrow{\Delta^*} K_{\Sigma_m}(X) \to R(\Sigma_m) \otimes K(X)$, now for

$f \in \mathrm{Hom}_{\mathbb{Z}}(R(\Sigma_m), \mathbb{Z})$, $R(\Sigma_m) \otimes K(X) \xrightarrow[f \otimes 1]{} \mathbb{Z} \otimes K(X) = K(X)$ gives

an operation on $K(X)$. In fact, operation of this types generates "all" the operations in K-theory. [Rezk, 2006] [M.F. Atiyah, 1966]