The Euler Characteristic Transform and Its Applications to Deep Learning

Euler characteristic For a simplicial complex K, we define the Euler characteristic χ as

$$\chi(K) = \sum_{n=0}^{\infty} (-1)^n \, |K_n|, \tag{1}$$

where $|K_n|$ denotes the cardinality of the set of n-simplices. The Euler characteristic is invariant under homeomorphisms and can be related to other properties of K.

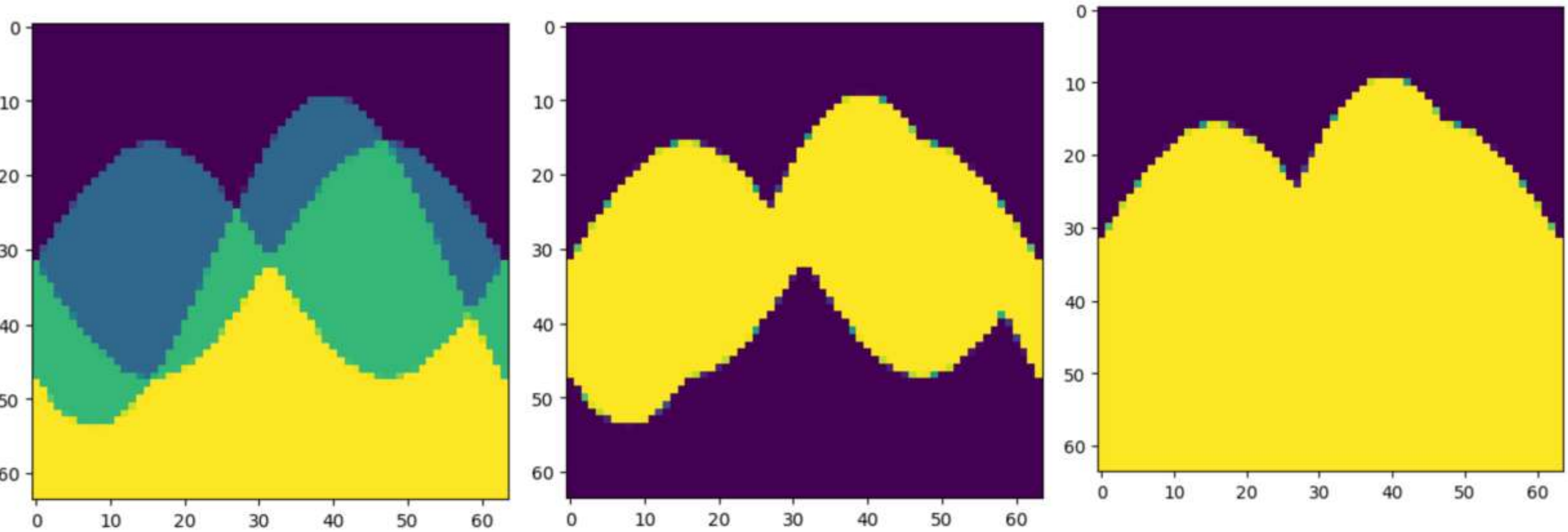
Filtrations The Euler characteristic $\chi(K)$ describes a simplicial complex K at a single scale. To obtain a multi-scale view, we equip K with a *filter function*. Let K be a simplicial complex and let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function. Restrict f to the vertex set (0-simplices) of K, and extend it to any simplex $\sigma \in K$ by

$$\tilde{f}(\sigma) := \max_{\tau \subseteq \sigma} \tilde{f}(\tau),$$

$$
\begin{aligned}
f\colon S^{n-1} \times \mathbb{R}^n &\to \mathbb{R} \\
(\xi, x) &\mapsto \langle x, \xi \rangle,
\end{aligned} \tag{3}
$$

where ξ is a *direction* (viewed as a point on a sphere of appropriate dimensionality), and $\langle \cdot, \cdot \rangle$ denotes the standard inner product. For a fixed ξ, we write $f_\xi := f(\xi, \cdot)$. Given a *height* $h \in \mathbb{R}$, we obtain a filtration of K by computing the preimage $f_\xi^{-1}((-\infty, h])$. The ECT is then defined as:

$$
\begin{aligned}
\mathrm{ECT}\colon S^{n-1} \times \mathbb{R} &\to \mathbb{Z} \\
(\xi, h) &\mapsto \chi\left(f_\xi^{-1}((-\infty, h])\right),
\end{aligned} \tag{4}
$$


0
10
20
30
40
50
60

$$
\begin{aligned}
\mathrm{ECT}\colon S^{n-1} &\times \mathbb{R} \to \mathbb{Z} \\
(\xi, h) &\mapsto \sum_{k}^{\dim K} (-1)^k \sum_{\sigma_k} \mathbb{1}_{[-\infty, h_\xi(\sigma_k))}(h),
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\mathrm{ECT}\colon S^{n-1} &\times \mathbb{R} \to \mathbb{Z} \\
(\xi, h) &\mapsto \sum_{k}^{\dim K} (-1)^k \sum_{\sigma_k} S\left(\lambda\left(h - h_\xi(\sigma_k)\right)\right),
\end{aligned} \tag{6}
$$

- For the synthetic data sets, we add DECT as the first layer of an MLP with 3 hidden layers. Each
- hidden layer has 25 units.

Table 1: MNIST point-cloud classification with $D = 2$ directions and $R = 16$ resolutions. Report mean accuracy (%) $\pm$ std over repeated runs.

Setting	Accuracy (%)
Fixed directions (Ours)	75.05 ± 0.45
Learnable directions (Ours)	81.97 ± 2.53
Fixed directions (Röell et al.,2024)	77.61 ± 7.98
Learnable directions (Röell et al.,2024)	81.29 ± 3.39

For a simplicial complex K and a function $g : K \to \mathbb{N}$, we define the *weighted Euler characteristic*

$$\chi^w(K, g) = \sum_{d=0}^{\dim(K)} (-1)^d \sum_{\sigma \in K^d} g(\sigma).$$

$$\tau < \sigma \;\Rightarrow\; g(\tau) \geq g(\sigma).$$

Algorithm 1 Grayscale Image to Weighted Complex

function IMAGETOWEIGHTEDCOMPLEX(A)
  ▷ $A \in \mathbb{N}^{m \times n}$ greyscale image matrix
  $V_{center} = \text{FIND}(A \neq 0)$
  ▷ treat nonzero pixels as coords for vertices
  $V = V_{center}$ ▷ initialize vertex list
  for $v \in V_{center}$ **do** ▷ add corner vertices
    append $v + [\pm 1/2, \pm 1/2]$ to V
  end for
  $V = \text{UNIQUE}(V)$ ▷ remove duplicates
  $F = []$ ▷ initialize face list
  for $v \in V_{center}$ **do**
    append triangles containing v to F
  end for
  $E =$ all resulting edges
  for $f \in F$ containing $v \in V_{center}$ **do**
    $Fw(f) =$ weight of corresponding pixel value
  end for
  for $v \in V$ **do**
    $Vw(v) =$ largest weight of face containing v
  end for
  for $e \in E$ **do**
    $Ew(e) =$ largest weight of face containing e
  end for
  return V, E, F, Vw, Ew, Fw
end function

Table 1. SVM ten-fold classification performance of vectorized image, ECT and WECT representations for the MNIST digit data.

Representation	Classification Rate
Image $\mathbb{R}^{28\times 28}$	87.84 ± 1.42 %
ECT $\mathbb{R}^{25\times 50}$	89.88 ± 1.66 %
WECT $\mathbb{R}^{25\times 50}$	94.68 ± 1.57 %

```
seed=42 epoch=001 train_loss=0.8766 train_acc=0.6978 val_loss=0.8891 val_acc=0.6958 test_acc=0.7053 epoch_time=13.28s
seed=42 epoch=010 train_loss=0.3935 train_acc=0.8700 val_loss=0.4367 val_acc=0.8598 test_acc=0.8634 epoch_time=17.52s
seed=42 epoch=020 train_loss=0.2891 train_acc=0.8999 val_loss=0.3698 val_acc=0.8782 test_acc=0.8899 epoch_time=15.22s
seed=42 epoch=030 train_loss=0.2101 train_acc=0.9297 val_loss=0.3136 val_acc=0.9002 test_acc=0.9056 epoch_time=18.71s
seed=42 epoch=040 train_loss=0.2074 train_acc=0.9278 val_loss=0.3318 val_acc=0.8988 test_acc=0.9011 epoch_time=21.48s
seed=42 epoch=050 train_loss=0.1730 train_acc=0.9400 val_loss=0.3256 val_acc=0.9048 test_acc=0.9099 epoch_time=17.60s
seed=42 epoch=060 train_loss=0.1670 train_acc=0.9417 val_loss=0.3396 val_acc=0.9024 test_acc=0.9084 epoch_time=17.64s
```

```
seed=42 epoch=001 train_loss=1.6339 train_acc=0.3977 val_loss=1.6450 val_acc=0.3936 test_acc=0.4036 epoch_time=13.18s
seed=42 epoch=010 train_loss=1.3356 train_acc=0.5228 val_loss=1.3822 val_acc=0.5112 test_acc=0.5203 epoch_time=20.24s
seed=42 epoch=020 train_loss=1.0929 train_acc=0.6082 val_loss=1.1464 val_acc=0.5836 test_acc=0.6012 epoch_time=17.71s
seed=42 epoch=030 train_loss=1.0542 train_acc=0.6272 val_loss=1.1219 val_acc=0.6008 test_acc=0.6152 epoch_time=19.89s
```

Injectivity

- PERSISTENT HOMOLOGY AND EULER INTEGRAL TRANSFORMS

  by ROBERT GHRIST, RACHEL LEVANGER, AND HUY MAI

- HOW MANY DIRECTIONS DETERMINE A SHAPE AND OTHER SUFFICIENCY RESULTS FOR TWO TOPOLOGICAL TRANSFORMS

  by JUSTIN CURRY, SAYAN MUKHERJEE, AND KATHARINE TURNER

Stability

Theorem 3.1 (Dłotko & Gurnari [6] Proposition 2). *Given geometric simplicial complexes K_1, K_2, with filtrations $\mathcal{F}$ and $\mathcal{G}$ respectively, then the L_1 distance of the Euler Characteristic Curves between them is bounded by the Wasserstein distance between the filtrations.*

$$\| \mathrm{ECC}_{\mathcal{F}}(K_1) - \mathrm{ECC}_{\mathcal{G}}(K_2)\|_1 \leq 2W_{1,\infty}(\mathrm{Dgm}(\mathcal{F}(K_1)), \mathrm{Dgm}(\mathcal{G}(K_2))$$

This result gives a relationship between the ECC of a complex with a filtration and the persistence diagrams obtained from the same filtration. We can further restrict the filtration to be the height function in a direction, making it suitable for the ECT.

$$\| \mathrm{ECC}_{\nu}(f(K)) - \mathrm{ECC}_{\nu}(g(K))\|_1 \leq 2W_{1,\infty}(\mathrm{Dgm}(h_{\nu}^{f(K)}), \mathrm{Dgm}(h_{\nu}^{g(K)})).$$

$$
\begin{aligned}
&\sum_{z\in\mathbb{N}} \chi(g^{-1}([z,\infty))) \\
&\qquad = \sum_{z\in\mathbb{N}} \sum_{d=0}^{\dim(K)} (-1)^d \#\{\sigma \in K \mid g(\sigma) \geq z\}^d \\
&\qquad = \sum_{d=0}^{\dim(K)} (-1)^d \sum_{z\in\mathbb{N}} \#\{\sigma \in K^d \mid g(\sigma) \geq z\} \\
&\qquad = \sum_{d=0}^{\dim(K)} (-1)^d \sum_{z\in\mathbb{N}} \sum_{\sigma\in K^d} \mathbf{1}_{g(\sigma)\geq z} \\
&\qquad = \sum_{d=0}^{\dim(K)} (-1)^d \sum_{\sigma\in K^d} g(\sigma) = \chi^w(K,g).
\end{aligned}
$$

$$WECC_{\nu}^{f,g}(t) = \sum_{z=1}^{m} ECC_{\nu}(f(K^{(z)}))(t), \qquad WECC_{\nu}^{h,g}(t) = \sum_{z=1}^{m} ECC_{\nu}(h(K^{(z)}))(t).$$

$$\|WECC_{\nu}^{f,g} - WECC_{\nu}^{h,g}\|_1 \leq \sum_{z=1}^{m} \|ECC_{\nu}(f(K^{(z)})) - ECC_{\nu}(h(K^{(z)}))\|_1.$$