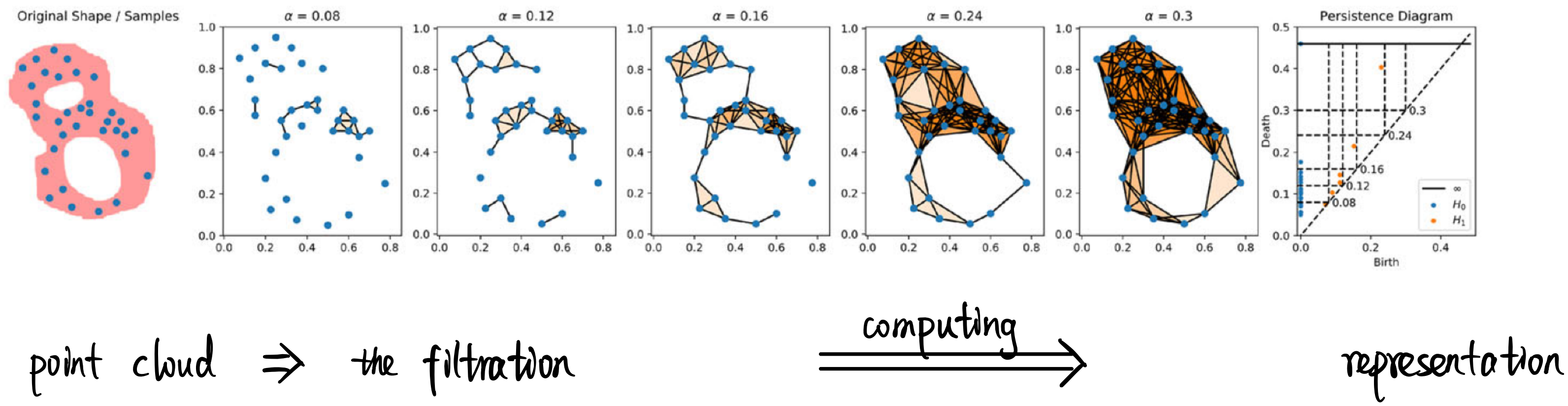
Interleaving

易思恒

TDA


Original Shape / Samples
α = 0.08
α = 0.12
α = 0.16
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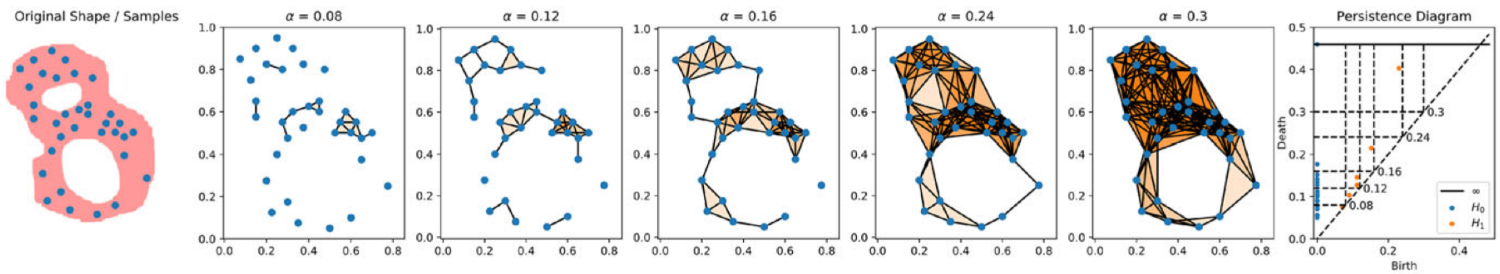
α = 0.3
Persistence Diagram
Death
Birth
H_0
H_1
point cloud ⇒ the filtration
computing
representation



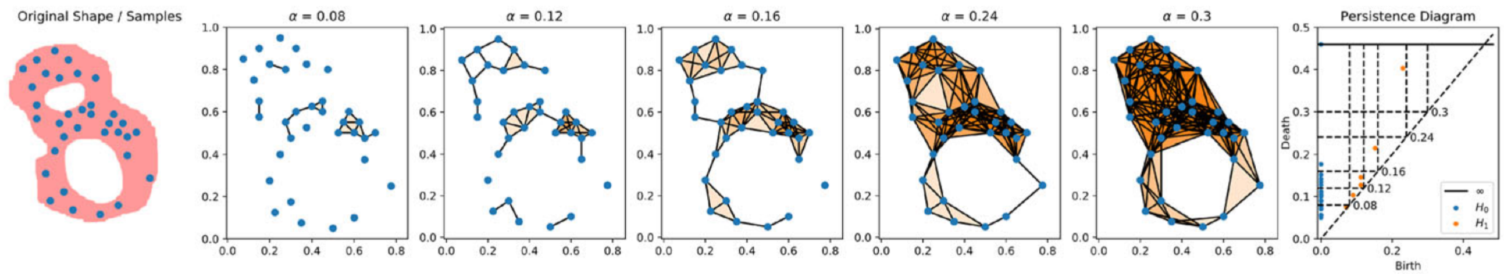
point cloud ⇒ the filtration $\xrightarrow{\text{computing}}$ representation

Vietoris-Rips Complex

Čech Complex

Dowker Complex

witness Complex



point cloud $\Rightarrow$ the filtration $\xrightarrow{\text{computing}}$ representation

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homology

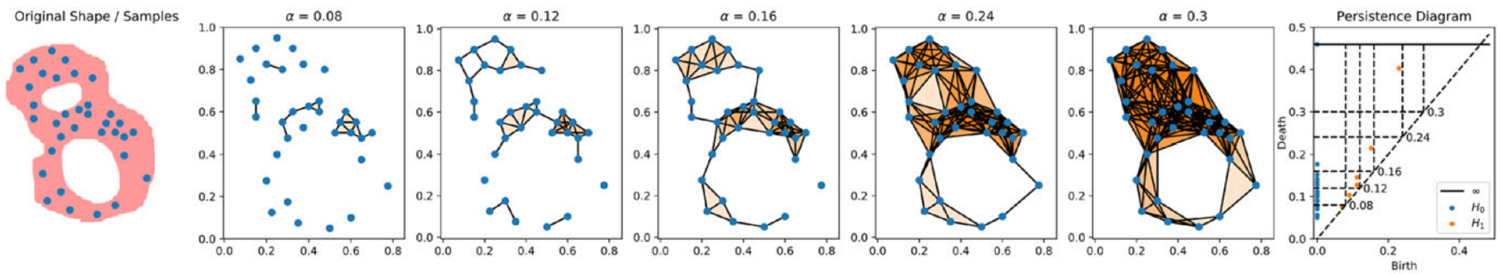
cohomology

$\mathcal{A}_\infty$-structure

mod-p Steenrod algebra module

(nomalized cochain complex) $\mathcal{E}_\infty$-structure

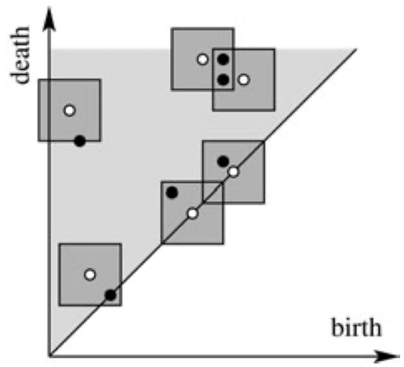
homotopy information



point cloud $\Rightarrow$ the filtration $\xRightarrow{\text{computing}}$ representation

the filtration	computing	representation
Vietoris-Rips Complex	homology	persistence diagram
Čech Complex	cohomology	barcode
Dowker Complex	$\mathcal{A}_\infty$-structure	landscape
witness Complex	mod-p Steenrod algebra module	persistent curve
	(nomalized cochain complex) $\mathcal{E}_\infty$-structure	persistent image.
	homotopy information	

describe the differences between different filtration by bottleneck distance.



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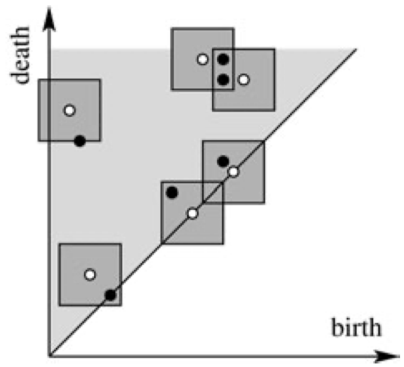

For all finite metric spaces P and Q,

$$d_B(\mathrm{Dgm}(H_i R(P)), \mathrm{Dgm}(H_i R(Q))) \leq d_{GH}(P, Q)$$

Gromov-Hausdorff distance.

More Theoretical

Abstracting

filtration $\rightsquigarrow$ an functor $X: I \to C$, I is a thin category C is a category.

Or an object $X \in ob(C^I)$

objects in $ob(C^I)$ are called persistence modules.

Abstracting

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example $H_*: Top \to gr\text{-}Vect$

$H^*: Top \to Alg$

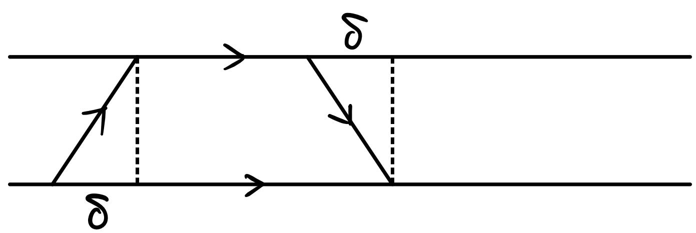
$\Pi^{Top}: Top \to Ho(Top)$

"distance" : extended pseudometric , interleaving distance.

δ-interleaving category , I^δ , is the thin category with object set $\mathbb{R}\times\{0,1\}$ and a morphism $(r,i)\to(s,j)$ if and only if either

(1) $r+\delta\leqslant s$ or

(2) $i=j$ and $r\leqslant s$.



$X,Y:\mathbb{R}\to C$, we say X and Y are δ-interleaved , if there is a functor $Z: I^\delta\to C$, s.t. $Z\circ E^0=X$ and $Z\circ E^1=Y$.

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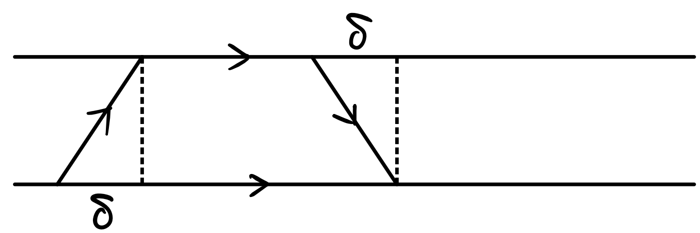

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interleaving distance $d_I(X,Y):=\inf\{\delta \mid X \text{ and } Y \text{ are } \delta\text{-interleaved}\}$

Multiplicative Interleaving Distance.

the interleaving distance of distinct representations

Theorem (Gregory Ginot and Johan Leray)

Let $\mathbb{P} = \{\text{prime numbers}\} \cup \{0\}$

(1) $$d_{\mathcal{E}_\infty} \geqslant d_{\mathbb{P}} \geqslant d_{\mathbb{P}_\infty} \begin{array}{l} \geqslant d_{\mathcal{A}_p\text{-}\mathcal{A}s} \geqslant \\ \geqslant d_{\mathcal{A}_\infty, \mathbb{F}_p} \geqslant \end{array} d_{\mathcal{A}s, \mathbb{F}_p} \geqslant d_{gr\text{-}Vect, \mathbb{F}_p}$$

for finite filtered data.

(2) More generally,

$$d_{\mathcal{E}_\infty} \geqslant d_{\mathcal{A}_p\text{-}ho\,Alg_{\mathcal{A}s}} \begin{array}{l} \geqslant d_{\mathcal{A}_p\text{-}\mathcal{A}s} \geqslant \\ \geqslant d_{ho\,Alg_{\mathcal{A}s}, \mathbb{F}_p} \geqslant \end{array} d_{\mathcal{A}s, \mathbb{F}_p} \geqslant d_{gr\text{-}Vect, \mathbb{F}_p}$$

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Note: $d_{gr\text{-}Vect, \mathbb{F}_p}(X, Y) = d_I(H_*(X; \mathbb{F}_p), H_*(Y; \mathbb{F}_p))$ $\quad$ $X, Y : \mathbb{R} \to Top$

$$= d_B(Dgm(H_*R(P)), Dgm(H_*R(Q)))$$

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Note: $d_{HI} \geq d_{\mathcal{E}_\infty}$

Homotopy Interleaving distance

Weak equivalence $\simeq$:

$X, Y : \mathbb{R} \to \mathrm{Top}$ or $X, Y \in \mathrm{Top}^{\mathbb{R}}$

Top is a model category, then $\mathrm{Top}^{\mathbb{R}}$ is a model category called projective model category.

$X \simeq Y$:

$$\begin{tikzcd}[column sep=small] & W_1 \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & \cdots \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & W_n \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ X & & W_2 & & W_{n-1} & & Y \end{tikzcd}$$

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Because all objects of Top is fibrant, all objects of $Top^{\mathbb{R}}$ is fibrant.

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Homotopy interleaving distance d_{HI}

δ-homotopy-interleaved : $X \simeq X'$, $Y \simeq Y'$ s.t. X' and Y' are δ-interleaved.

$d_{HI} := \inf \{ \delta \mid X, Y \text{ are } \delta\text{-homotopy-interleaved} \}$

Three elementary properties.

(1) Stable

① discrete. For all finite metric spaces P, Q,

$$d(\mathcal{R}(P), \mathcal{R}(Q)) \leq d_{GH}(P, Q)$$

② continuous. For any $T \in \mathrm{ob}\,\mathbf{Top}$ and functions $\gamma, \kappa : T \to \mathbb{R}$

$$d(S(\gamma), S(\kappa)) \leq d_\infty(\gamma, \kappa)$$

(2) homotopy invariant

If $X \simeq Y$, then $d(X, Y) = 0$.

(3) homology bounding

$$d_B(\mathrm{Dgm}(H_*\mathcal{R}(P)), \mathrm{Dgm}(H_*\mathcal{R}(Q))) \leq d(X, Y)$$

Claim: Any (continuous) stable and homotopy invariant distance on $\mathrm{ob}(\mathbf{Top}^I)$ satisfies discrete stability.

$d_{gr\text{-}Vect}$ satisfies (1) and (2).

Theorem (Andrew J. Blumberg and Michael Lesnick)

d_{HI} is a distance on $ob(Top^{\mathbb{R}})$ satisfying the stability, homotopy invariance and homology bounding property.

Theorem (Andrew J. Blumberg and Michael Lesnick)

d_{HI} is a distance on $ob(Top^{\mathbb{R}})$ satisfying the stability, homotopy invariance and homology bounding property.

d_{HI} satisfies the triangle inequality. ?

Given X and Y are δ-homotopy-interleaved, Y and Z are ε-homotopy-interleaved, prove that X and Z are $(\delta+\varepsilon)$-homotopy-interleaved.

Marked category $\mathcal{M}$: finit , thin , equipped with some extra informations.

Generated interleaving category $\bar{\mathcal{M}}$: $\mathrm{ob}\bar{\mathcal{M}} = \mathrm{ob}\mathcal{M} \times \mathbb{R}$, thin

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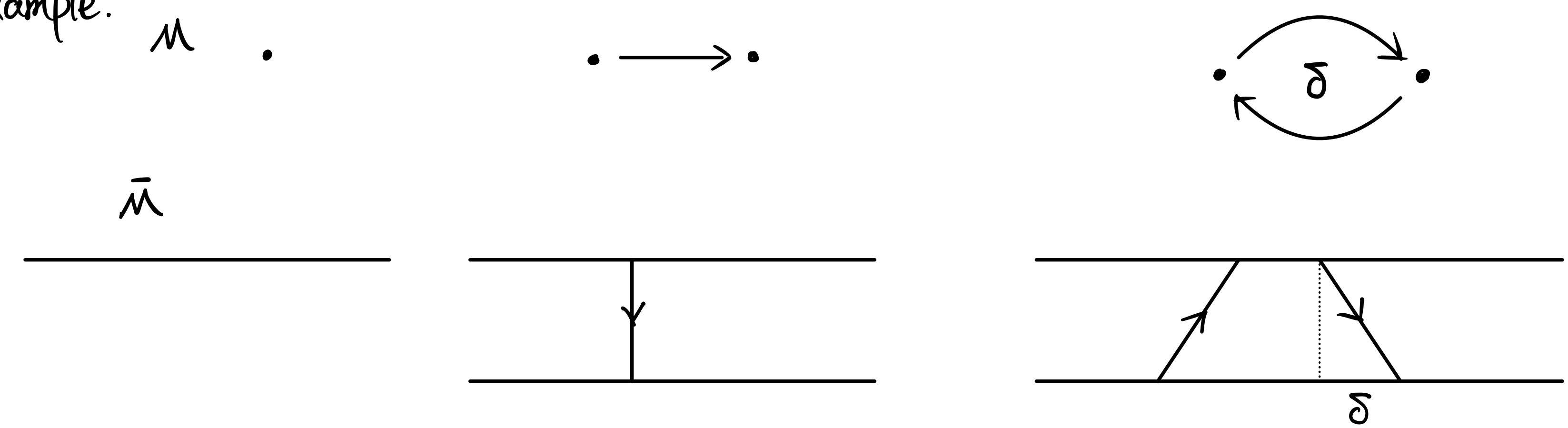
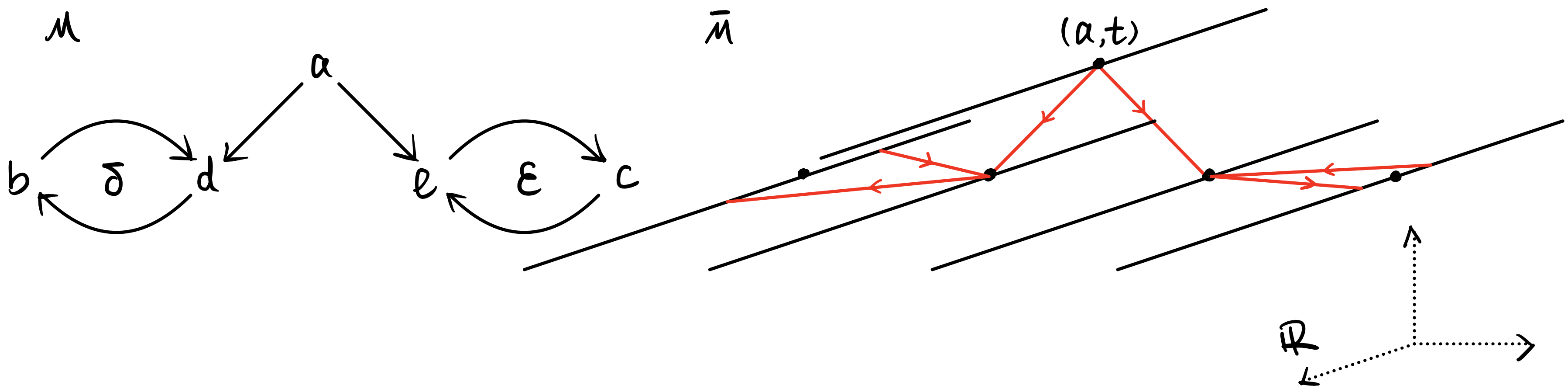
Example.

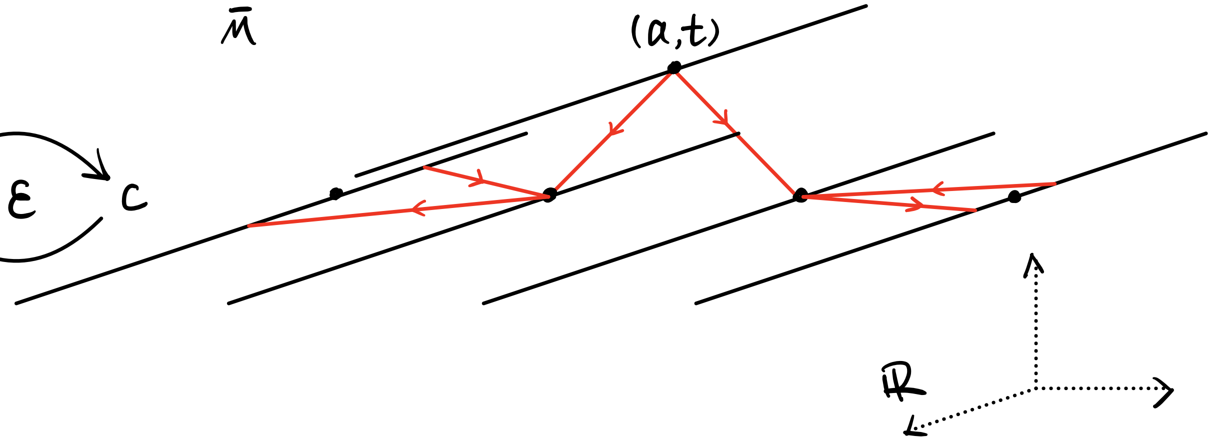




M
a
b
δ
d
e
ε
c
M̄
(a,t)
ℝ

$\mathcal{M}$

$$\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
b \arrow[r, bend left] \arrow[r, phantom, "\delta"] & d \arrow[l, bend left] & & e \arrow[r, bend left] \arrow[r, phantom, "\varepsilon"] & c \arrow[l, bend left]
\end{tikzcd}$$



$F : \bar{\mathcal{M}} \longrightarrow \mathrm{Top}$ $\quad$ $F(a)_t := F(a,t)$ then $F(a)$ is an object of $\mathrm{Top}^{\mathbb{R}}$.

Similarly, $F(b)$, $F(c)$ $\cdots$

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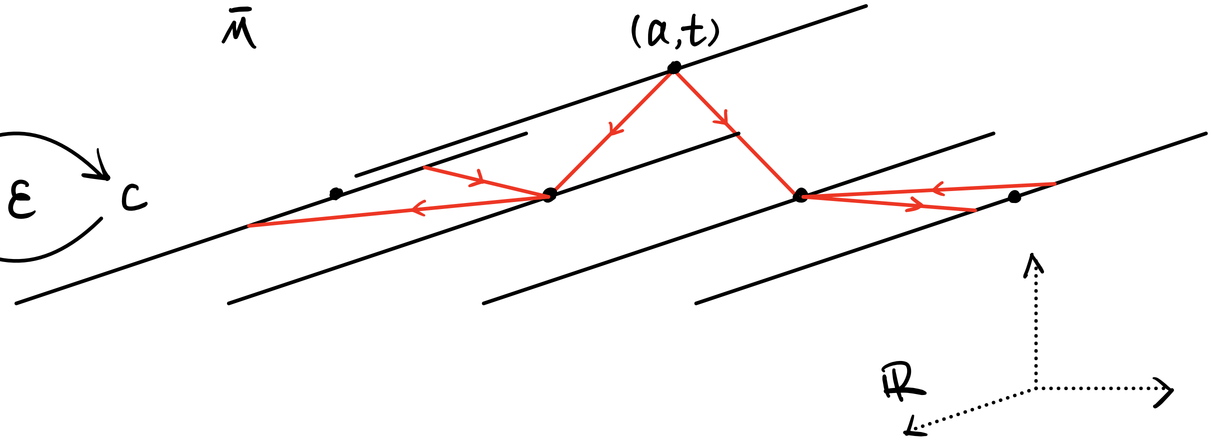

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Similarly, $F(b), F(c)$ $\cdots$

WLOG, we may assume that F is cofibrant by taking a cofibrant replacement of F.

describe that X and Y are δ-homotopy-interleaved, Y and Z are ε-homotopy-interleaved.

M :

$$\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
b \arrow[r, bend left] \arrow[r, phantom, "\delta"] & d \arrow[l, bend left] & & e \arrow[r, bend left] \arrow[r, phantom, "\varepsilon"] & c \arrow[l, bend left]
\end{tikzcd}$$

$F: \bar{M} \longrightarrow Top$ s.t. $F(b) \simeq X$ $\quad F(c) \simeq Z$ $\quad F(d) \simeq F(e) \simeq Y$

$F(a) \xrightarrow{\simeq} F(d)$ $\quad F(a) \xrightarrow{\simeq} F(e)$

F :

$$\begin{tikzcd}
 & & & F(a) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & & \\
X \arrow[r, phantom, "\simeq"] & F(b) \arrow[r, bend left] \arrow[r, phantom, "\delta"] & F(d) \arrow[l, bend left] \arrow[r, phantom, "\simeq"] & Y \arrow[r, phantom, "\simeq"] & F(e) \arrow[r, bend left] \arrow[r, phantom, "\varepsilon"] & F(c) \arrow[l, bend left] \arrow[r, phantom, "\simeq"] & Z
\end{tikzcd}$$

prove that X and Z are $(\delta+\varepsilon)$-homotopy-interleaved.

$\mathcal{N}$

$$\begin{tikzcd}
 & & & a \arrow[dl] \arrow[dr] & & & \\
b \arrow[rr, bend left] \arrow[rr, phantom, "\delta"] \arrow[dr, dashed] & & d \arrow[ll, bend left] \arrow[dr, dashed] & & e \arrow[rr, bend left] \arrow[rr, phantom, "\varepsilon"] \arrow[dl, dashed] & & c \arrow[ll, bend left] \arrow[dl, dashed] \\
 & f \arrow[rr, dashed, bend left] \arrow[rr, phantom, "\delta"] & & h \arrow[ll, dashed, bend left] \arrow[rr, dashed, bend left] \arrow[rr, phantom, "\varepsilon"] & & g \arrow[ll, dashed, bend left] &
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{\mathcal{M}} \arrow[r, "F"] \arrow[d, "\iota"'] & Top \\
\bar{\mathcal{N}} \arrow[ur, dashed, "\widetilde{F}"'] &
\end{tikzcd}$$

If we can find out an extension $\widetilde{F}: \bar{\mathcal{N}} \to Top$ s.t. $\widetilde{F}(b) \xrightarrow{\simeq} \widetilde{F}(f)$ and $\widetilde{F}(c) \xrightarrow{\simeq} \widetilde{F}(g)$, Done.

$$\begin{tikzcd}
 & & & \widetilde{F}(a) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & & \\
X \simeq \widetilde{F}(b) \arrow[rr, bend left] \arrow[rr, phantom, "\delta"] \arrow[dr, dashed, "(\text{if } \simeq)"'] & & \widetilde{F}(d) \simeq \arrow[ll, bend left] \arrow[dr, dashed] & Y & \simeq \widetilde{F}(e) \arrow[rr, bend left] \arrow[rr, phantom, "\varepsilon"] \arrow[dl, dashed] & & \widetilde{F}(c) \simeq Z \arrow[ll, bend left] \arrow[dl, dashed, "(\text{if } \simeq)"] \\
 & (\text{then } X \simeq)\ \widetilde{F}(f) \arrow[rr, dashed, bend left] \arrow[rr, phantom, "\delta"] & & \widetilde{F}(h) \arrow[ll, dashed, bend left] \arrow[rr, dashed, bend left] \arrow[rr, phantom, "\varepsilon"] & & \widetilde{F}(g)\ (\text{then } \simeq Z) \arrow[ll, dashed, bend left] &
\end{tikzcd}$$

left Kan-extension.

$$
\begin{tikzcd}
\bar{M} \arrow[r, "F"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathrm{Top} \\
\bar{N} \arrow[ur, "\mathrm{Lan}_\iota F"'] &
\end{tikzcd}
$$

$\widetilde{F} := \mathrm{Lan}_\iota F$

$$
\begin{tikzcd}
 & & & \widetilde{F}(a)_t \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & & \\
X_{t\pm\delta} \simeq & \widetilde{F}(b)_{t\pm\delta} \arrow[r, bend left] \arrow[dr, "\simeq"'] & \widetilde{F}(d)_t \arrow[l, bend left] \arrow[l, phantom, "\delta"] \arrow[dr, "\simeq"] & & \widetilde{F}(e)_t \arrow[r, bend left] \arrow[dl, "\simeq"'] & \widetilde{F}(c)_{t\pm\varepsilon} \arrow[l, bend left] \arrow[l, phantom, "\varepsilon"] \arrow[dl, "\simeq"] & \simeq Z_{t\pm\varepsilon} \\
 & & \widetilde{F}(f)_{t\pm\delta} \arrow[r, bend left] & \widetilde{F}(h)_t \arrow[l, bend left] \arrow[l, phantom, "\delta"] \arrow[r, bend left] & \widetilde{F}(g)_{t\pm\varepsilon} \arrow[l, bend left] \arrow[l, phantom, "\varepsilon"] & &
\end{tikzcd}
$$

Universality.

Fact: For any directed set I, each cofibrant diagram in Top^I is 1-critical.

$X: I \to \mathrm{Top}$ is 1-critical : X is closed filtration and for each $\gamma \in \operatorname{colim} X$ the set $\{a \in I \mid \gamma \in \operatorname{im} \mu_a^X\}$ has a minimum element.

Obviously, if X is 1-critical, then we have a function

$$\zeta^X : \operatorname{colim} X \to I.$$

Proposition. For any δ-interleaved $\mathbb{R}$-spaces X, Y, there exists a topological space T and functions $\gamma^X, \gamma^Y : T \to \mathbb{R}$ such that $S(\gamma^X) \simeq X$, $S(\gamma^Y) \simeq Y$ and $d_\infty(\gamma^X, \gamma^Y) \leq \delta$.

Theorem (Andrew J. Blumberg and Michael Lesnick)

If d is any stable and homotopy invariant distance on $\mathbb{R}$-spaces, then $d \leq d_{HI}$.

persistent Whitehead Conjecture

Theorem (Whitehead theorem for model categories)

For any model category C, a weak equivalence between cofibrant-fibrant objects in C is a homotopy equivalence.

Persistent Whitehead Conjecture. (Andrew J. Blumberg and Michael Lesnick)

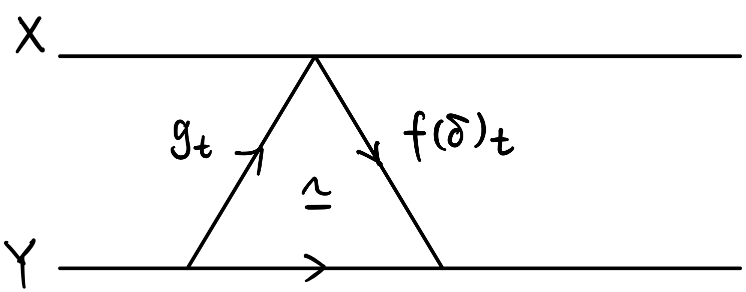
the internal maps $\{X_{r,r+\delta}\}_{r\in\mathbb{R}}$ assemble into a morphism $\varphi^{X,\delta}: X \to X(\delta)$

δ-homotopy equivalences.

Given $\mathbb{R}$-spaces X and Y, we will say a pair of morphisms $f: X \to Y(\delta)$ and $g: Y \to X(\delta)$ are (inverse) δ-homotopy equivalence if

$$g(\delta)\circ f \simeq \varphi^{X,2\delta} \quad \text{and} \quad f(\delta)\circ g \simeq \varphi^{Y,2\delta}$$

where $f(\delta): X(\delta) \to Y(2\delta)$ is the map induced by f, and is $g(\delta)$ defined analogously.



naive version 1.

For X and Y connected cofibrant $\mathbb{R}$-spaces, $\delta \geq 0$, and morphism $f: X \to Y(\delta)$ with $\pi_i f : \pi_i X \to \pi_i Y(\delta)$ a δ-interleaving morphism for all i, f is a δ-homotopy equivalence.

naive version 2.

Given X, Y and f as in the previous conjecture, X and Y are δ-homotopy-interleaved.

Example. X' is trivial, i.e. $X'_r = *$ for all r.

$$Y^n : \mathbb{R} \to CW$$

$$Y^n_r := \begin{cases} \underbrace{\mathbb{S}^{2^i} \times \cdots \times \mathbb{S}^{2^i}}_{2^{n-i} \text{ copies}} & \text{for } r \in [2i, 2i+2) \quad i \in \{0, 1, \cdots, n\} \\ * & \text{for } r \in (-\infty, 0) \cup [2n+2, +\infty) \end{cases}$$

$$Y^n_r \to Y^n_s \qquad r \in [2i, 2i+2) \quad s \in [2i+2, 2i+4) \quad i \in \{0, 1, \cdots, n-1\}$$

$$\underbrace{\mathbb{S}^{2^i} \times \cdots \times \mathbb{S}^{2^i}}_{2^{n-i} \text{ copies}} \longrightarrow \underbrace{\mathbb{S}^{2^{i+1}} \times \cdots \times \mathbb{S}^{2^{i+1}}}_{2^{n-i-1} \text{ copies}}$$

is induced by $\mathbb{S}^{2^i} \times \mathbb{S}^{2^i} \to \mathbb{S}^{2^i} \times \mathbb{S}^{2^i} / \mathbb{S}^{2^i} \vee \mathbb{S}^{2^i} = \mathbb{S}^{2^{i+1}}$

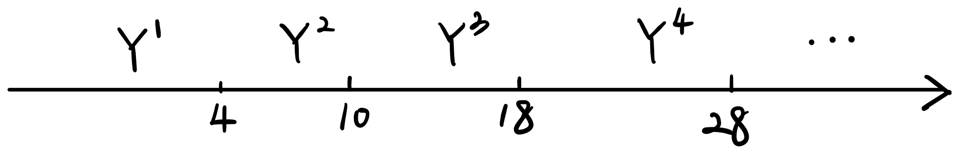
example. $Y^3_r = \underline{\underline{\underline{\mathbb{S}^2 \times \mathbb{S}^2}}} \times \underline{\underline{\underline{\mathbb{S}^2 \times \mathbb{S}^2}}}$ $\quad r \in [2, 4)$

$Y^3_s = \mathbb{S}^4 \times \mathbb{S}^4$ $\quad s \in [4, 6)$

By the long exact sequence of homotopy group, $\pi_* Y^n_{r, r+2}$ is trivial for all r. Then $X' \to Y^n(1)$ and $Y^n \to X'(1)$ induce 1-interleavings on all based persistent homotopy groups.

Obviously, X' and Y^n are not δ-homotopy equivalence for any $\delta < n+1$ by checking celluar their homology.

Define Y'



$X = QX \qquad Y = QY \qquad f : X \longrightarrow Y(1) \qquad f = Q$ (trivial map).

Version 3.

Suppose we are given connected cofibrant $\mathbb{R}$-spaces $X, Y : \mathbb{R} \to CW$ with each X_r and Y_r of dimension at most d, and $f : X \to Y(\delta)$ with $\pi_i f : \pi_i X \to \pi_i Y(\delta)$ a δ-interleaving morphism for all i. Then there is a constant $c \geq 1$, depending only on d, such that

(i) the map induced by f is a $c\delta$-homotopy equivalence,

(ii) X and Y are $c\delta$-homotopy-interleaved.

Ref.

Andrew J. Blumberg and Micheal Lesnick. Universality of the homotopy interleaving distance, 2022.

Gregory Ginot and Johan Leray. Multiplicative persistent distances, 2021.

W. G. Dwyer and J. Spalinski. Homotopy theories and model categories. Handbook of algebraic topology.