M is a finitely generated n-graded A_n-module

$\rho(M) =: \mathbb{k} \otimes_{A_n} M \qquad \xi(M) =: \xi(\rho(M))$

minimal free resolution $\quad \cdots \to F_1 \to F_0 \to M$

$\xi_0(M) =: \xi(F_0) \quad \xi_1(M) =: \xi(F_1)$

$S(\xi_0, \xi_1) = \{$ n-graded A_n-submodules L of $F \mid \xi(\rho(L)) = \xi_1 \}$ $\quad (\xi(F) = \xi_0)$

$G_F =: \mathrm{Aut}(F)$

$I(\xi_0, \xi_1) = \{$ isomorphic classes of f.g. n-graded A_n-modules $M \mid \xi_0(M) = \xi_0,$
$\xi_1(M) = \xi_1 \}$

$$\begin{aligned} q : S(\xi_0, \xi_1) &\longrightarrow I(\xi_0, \xi_1) \\ L &\longmapsto [F/K] \end{aligned}$$

Thm (Complete Classification)

Let F be as above. The map q satisfies the formula $q(g \cdot L) = q(L)$ $g \in G_F$ and consequently indues a map $\bar{q} : G_F \backslash S(F, \xi_1) \to I(\xi_0, \xi_1)$. Moreover, $\bar{q}$ is bijective.

$G_F \curvearrowright S(\xi_0, \xi_1)$

$$\mathrm{ARR}_{\xi,\delta}(F) \subseteq \mathcal{E} = \coprod \mathrm{Gr}_{\delta(v)}(F_v)$$

$$\Downarrow$$

the constructure of $\mathcal{E}$

$\Downarrow$ additional condition

$$\mathrm{ARR}_{\xi,\delta}(F)$$

Parameterazation.

GOAL : parameterize $S(\xi_0, \xi_1)$

Let $\xi = (\overset{\text{gap}}{V}, \alpha)$ denote any multiset, and let $\delta : V \to \mathbb{Z}$ be any function.

For any f.g. free n-graded A_n-module F, let $ARR_{\xi,\delta}(F)$ denote the set of all assignments $v \longrightarrow L_v$, where $v \in V$, and L_v is a $\mathbb{k}$-linear subspace of F_v, which satisfy the following three conditions :

(i). $v' \lesssim v \Rightarrow x^{v-v'} L_{v'} \subseteq L_v$

(ii). $\dim_{\mathbb{k}}(L_v) = \delta(v)$

(iii). $\dim \left(L_v / \sum_{v' \lnsim v} x^{v-v'} L_{v'} \right) = \alpha(v)$ for all $v \in V$.

The goal is to demonstrate that $ARR_{\xi,\delta}(F)$ is in bijective correspondence with the set of points of a quasi-projective variety over the field $\mathbb{k}$.

(这里只给出了部分子模的参数化)

1. $ARR_{\xi,\delta}(F) \subseteq \mathcal{E} =: \prod_{v \in V} Gr_{\delta(v)}(F_v)$

(ξ,δ)-frame for F : a family of linear embeddings $\{j_v : W_v \to F_v\}_{v \in V}$

$W_v \cong \mathbb{k}^{\delta(v)}$

The (ξ,δ)-frame determines a family of subspaces $L_v = \mathrm{Im}(j_v)$

$\mathcal{F}(F) =: \{(\xi,\delta)\text{-frames}\}$

$GL(W_v) \cong GL_{\delta(v)}(\mathbb{k}) \Rightarrow \Gamma =: \prod_{v \in V} GL(W_v) \cong \prod_{v \in V} GL_{\delta(v)}(\mathbb{k})$

$\Gamma \curvearrowright \mathcal{F}(F)$: $\sigma_v \in GL(W_v)$ $\{\sigma_v\} \cdot \{j_v\} = \{j_v \circ \sigma_v\}$

$\Rightarrow$ The orbit space of this action is the set of all families of subspaces $\{L_v\}_{v \in V}$ s.t. $\dim_{\mathbb{k}}(L_v) = \delta(v)$ for all $v \in V$ i.e. $\mathcal{E}$.

2. frames $\rightsquigarrow$ matrices

Given $F \cong \bigoplus_{i=1} A_n(v_i)$

For V, we can choose a enough large $v^* \in \mathbb{Z}^n$, s.t. $V \lesssim v^*$.

For any $v \in V$, $F_v \cong G_v \subseteq F_{v^*}$ and $x^{v^*-v} \cdot F_v = G_v$

In $\mathrm{ARR}_{\xi,\delta}(F)$, $L_v \subseteq F_v$ are identified $L_v^* \subseteq G_v \subseteq F_{v^*}$

(i)′ $L_v^* \subseteq G_v$ for all v

(ii)′ If $v \lesssim v'$, then $L_v^* \subseteq L_{v'}^*$

(iii)′ $\dim_{\mathbb{k}}(L_v^*) = \delta(v)$

(iv)′ $\dim_{\mathbb{k}}(L_v^* / \sum_{v' \precnsim v} L_{v'}^*) = \alpha(v)$ for all $v \in V$.

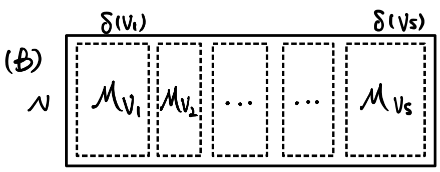
Let e_i denote the generator for the summand $A_n(v_i)$. And let $\mathcal{B} = \{x^{v^*-v_i} e_i\}_{i=1,\cdots,N}$ and $\mathcal{B}_v = \{x^{v^*-v_i} e_i \mid v_i \lesssim v\}$ is a basis for G_v.

$\mathcal{B}(v) =: \mathcal{B}_v - \bigcup_{v' \precnsim v} \mathcal{B}_{v'} \Rightarrow \mathcal{B} = \bigsqcup_{v \in V} \mathcal{B}(v)$

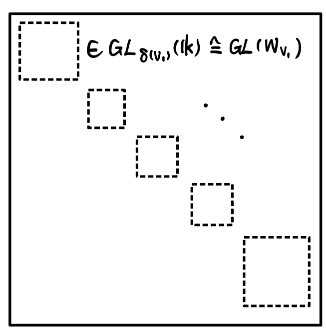
对每个 $\mathcal{B}(v)$，我们给定一个序

Similarly, we also can decompose the basis $\bigoplus_{v \in V} W_v$ as $\mathcal{B} = \bigsqcup_{v \in V} \mathcal{B}(v)$, $\mathcal{B}(v)$ consisting of the basis elements for the copy of W_v corresponding to v.

$(\mathcal{B})$ the block corresponding to $\mathcal{B}(v)$ and $\mathcal{B}(v)$ is identically zero if $v \not\lesssim v'$



$\longleftrightarrow$ a (ξ, δ)-frame



$(N = \dim(F_{v^*}) = \#\mathcal{B})$

$M = (M_{v_1} | \cdots | M_{v_s})$

The group action described above of the group $\prod_{v \in V} GL(W_v)$ on the set of frames can be interpreted as multiplication on the right by $\prod_{v \in V} GL_{\delta(v)}(\mathbb{k})$ on the corresponding matrix.

$\mathcal{E}$ = the orbit space $\Gamma \curvearrowright \mathcal{F}(F)$

= the orbit space $\Gamma \curvearrowright$ the quasiprojective variety

$\boxed{\text{all } N\times D \text{ matrices so that the rank of each submatrix } M_v \text{ is full i.e. } \delta(v) \text{ and so that the block of the matrix } M_v \text{ corresponding the subset } B(v')\subseteq B \text{ is identically zero whenever } v'\nleq v.}$

Geometry invariant theory

the action has closed orbits and satisfies the stability hypothesis

$\Rightarrow$ the action admits a geometric quotient

$\Rightarrow$ the set of orbits is naturally a quasiprojective variety.

3.

(i)'' $L_v^* \subseteq G_v$ for all v : ~~the block~~ the blocks of M_v corresponding to the set $B(v')$ is zero if $v'\nleq v$. (is already satisfied)

(ii)'' If $v \lneq v'$, then $L_v^* \subseteq L_{v'}^*$: rank$(M(v,v')) = \delta(v')$

$(M(v,v') = [M_v \mid M_{v'}])$

$\Leftrightarrow$ all $(\delta(v')+1)\times(\delta(v')+1)$ minors of $M(v,v')$ vanish.

(iii)'' $\dim_{\mathbb{k}}(L_v^*) = \delta(v)$: rank$(M_v) = \delta(v)$ (is already satisfied)

(iv)'' $\dim(L_v^* / \sum_{v'\lneq v} L_{v'}^*) = \alpha(v)$ for all $v\in V$: rank$(\lambda(v)) = \delta(v)-\alpha(v)$

($\lambda(v) = [M_{v_1'} \mid M_{v_2'} \mid \cdots \mid M_{v_j'}]$ where $\{v_1', \cdots, v_j'\}$ is an enumeration of all v_i' for $v_i' \lneq v$)

$\Leftrightarrow$ the set for which all the $(\delta(v)-\alpha(v)+1)\times(\delta(v)-\alpha(v)+1)$ minors of $\lambda(v)$ vanish, and removing from it the set for which all $(\delta(v)-\alpha(v))\times(\delta(v)-\alpha(v))$ minors vanish.

4. Let $\mathcal{F}_0(F) \subseteq \mathcal{F}(F)$ denote the quasiprojective variety of all matrices

satisfying the conditions (i)″ ~ (iv)″

Obviously, $\mathcal{F}_0(F) \overset{1-1}{\longleftrightarrow} ARR_{\xi,\delta}(F)$

Example.

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\Bbbk & \bullet & \bullet & \bullet & \bullet \\
\bullet & \Bbbk & \bullet & \bullet & \bullet \\
\bullet & \bullet & \Bbbk & \bullet & \bullet \\
\Bbbk^2 & \bullet & \bullet & \Bbbk & \bullet
\end{array}$$

$\xi_0 = \{((0,0),2)\}$ $\quad$ $\xi_1 = \{((3,0),1),\ ((2,1),1),\ ((1,2),1),\ ((0,3),1)\}$

$(0,0)$: the complex of two loops.

$(3,0), (2,1), (1,2), (2,1)$ we choose a surface to sew between the two loop such that no two complexes are sewn the same.

1. Classification.

$F(\xi_0) = A_2 \oplus A_2 \qquad GL(F(\xi_0)) = GL_2(\Bbbk)$

For $\forall (v,i) \in \xi_1$, $\dim F(\xi_0)_v = 2$, $\dim F(\xi_1)_v = 1$

$\Rightarrow Gr_{\dim F(\xi_1)_v} F(\xi_0)_v = Gr_1(\Bbbk^2) = \mathbb{P}^1(\Bbbk)$

$\Rightarrow$ Classification : the orbit space of $GL_2(\Bbbk) \curvearrowright \mathbb{P}^1(\Bbbk)^4$.

(the action is evident)

2. Consider subspace Ω of the orbit space containing pairwise-distinct lines

$GL_2(\Bbbk) \curvearrowright \{(l_1, l_2, l_3, l_4) \in \mathbb{P}^1(\Bbbk)^4 \mid l_i \neq l_j \text{ for } i \neq j\}$

Using matrices from $GL_2(\Bbbk)$, we can transform the lines so that :

(i) l_1 becomes the x-axis.

(ii) l_2 becomes the y-axis.

(iii) l_3 become the $\{x=y\}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} - \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ - \end{bmatrix} \quad \Rightarrow \text{ determine } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ up to multiplying constant}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$$

Then $(l_1, l_2, l_3, l_4) \xrightarrow{GL_2(\Bbbk)} (x\text{-axis},\ y\text{-axis},\ \{y=x\},\ l_4')$

$\Rightarrow \quad \Omega_2 \xleftrightarrow{1-1} \mathbb{P}^1(\Bbbk) - \{0, 1, \infty\} = \Bbbk - \{0, 1\}$.

If $\Bbbk = \mathbb{F}_p$ $\cdots$

If $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ $\cdots$

(1) It doesn't matter where the generators is, then $S(\xi_0, \xi_1)$ is the set $\{L \text{ is the submodule of } F \mid \xi(L) = \xi_1\ \boxed{\text{and } \xi(F/L) = \xi(F) = \xi_0}\ \}$.

(2) The locations of gaps are also unrelated, then $ARR_{\xi,\delta}(F) = S(\xi_0, \xi_1)$

i.e. $ARR_{\xi,\delta}(F)$ describes all of $S(\xi_0, \xi_1)$, not a part of $\mathcal{B}$.

And the three conditions about $ARR_{\xi,\delta}(F)$ can be simplified to

$\dim_{\Bbbk}(L_v) = \delta(v) = \alpha(v)$, so $ARR_{\xi,\delta}(F) = \mathcal{E} = \mathbb{P}^1(\Bbbk)^4$.

$$F: \quad y\uparrow \quad \begin{array}{ccc} \bullet & \bullet & \bullet \\ c,d & \bullet & \bullet \\ a,b & e,f & \bullet \\ & \overrightarrow{x} & \end{array} \qquad \xi = \{((0,0),2),\ ((0,1),2),\ ((1,0),2)\}$$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ y\cdot a & x\cdot c & \bullet \\ \bullet & x\cdot b & \bullet \end{array} \qquad \xi = \{((0,1),1),\ ((1,0),1),\ ((1,1),1)\}$$

$$\begin{cases} \dim(L_1)_{(1,1)} = 3 \\ \dim(L_2)_{(1,1)} = 2 \end{cases} \quad (ARR \neq S(\xi_0, \xi_1))$$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ y\cdot a & x\cdot c & \bullet \\ \bullet & x\cdot a & \bullet \end{array} \qquad \xi = \{((0,1),1),\ ((1,0),1),\ ((1,1),1)\}$$

$$\begin{cases} \xi_0(F/L_2) = \xi(F) \\ \xi_0(F/L_3) \neq \xi(F) \end{cases} \quad (q \text{ is not well-defined})$$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ c & y\cdot f & \bullet \\ \bullet & e & \bullet \end{array} \qquad \xi = \{((0,1),1),\ ((1,0),1),\ ((1,1),1)\}$$