One-dimensional Persistence

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots \to$$
$$\Downarrow H_k$$
$$H_k(X_0) \to H_k(X_1) \to H_k(X_2) \to \cdots \to H_k(X_n) \to \cdots$$

A graded module M over a graded ring R is a module equipped with a directed decomposition, $M = \bigoplus_i M_i$ $i \in \mathbb{Z}$, s.t. the action of R on M is defined by bilinear pairings $R_n \otimes M_m \to M_{n+m}$.

$M = \bigoplus_{i \geq 0} H_k(X_i)$ is a graded $R[t]$-module.

Specially, we consider graded $\Bbbk[t]$-module M where $\Bbbk$ is a field.

Def. A persistence module M is a family of R-modules M_i, together with homomorphisms $\varphi_i : M_i \to M_{i+1}$

Def. A persistence module $\{M_i, \varphi_i\}$ is of finite type if each component is finitely generated R-module, and φ_i are isomorphisms for $i \geq m$ for some integer m.

Correspondence:

$\{M_i, \varphi_i\}_{i \geq 0}$ is a persistence module over R. We equip $R[t]$ with the standard grading and define a graded module over $R[t]$ by $\alpha(M) = \bigoplus_{i \geq 0} M_i$, and the action of is given by

$$t \cdot (m_0, m_1, m_2, \cdots) = (0, \varphi_0(m_0), \varphi_1(m_1), \varphi_2(m_2), \cdots)$$

Thm the category of persistence modules of finite type over R

$$\cong$$

the category of finitely generated non-negatively graded modules over $R[t]$

Classification:

Consider persistence modules of finite type over $\mathbb{k}$.

Because $\mathbb{k}[t]$ is PID, $M \cong \left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} \mathbb{k}[t]\right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} \mathbb{k}[t]/(t^{n_j})\right)$

(Σ^{α} denotes α-shift upward in grading)

Parameterization:

$\Sigma^{a} \mathbb{k}[t] \rightsquigarrow [a, +\infty)$

$\Sigma^{b} \mathbb{k}[t]/(t^{c}) \rightsquigarrow [b, b+c)$

Multidimensional Persistence

finite type.

Def $\vec{u}, \vec{v} \in \mathbb{N}^n$, $\vec{u} \lesssim \vec{v}$ if $u_i \le v_i$ all i

multiset is a set within which an element may appear multiple times.

$A_n =: \mathbb{k}[x_1, \cdots, x_n]$ is a n-graded ring, A_n is graded by

$A_v = \vec{x}^{\vec{v}}$ $\quad \vec{x} = (x_1, \cdots, x_n)$ $\quad \vec{v} = (v_1, \cdots, v_n)$ $\quad \vec{x}^{\vec{v}} = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$.

an n-graded module over an n-graded module over an n-graded ring R is an Abelian group M equipped with a decomposition $M \cong \bigoplus_{\vec{v}} M_{\vec{v}}$, $\vec{v} \in \mathbb{N}$ together with a R-module structure so that $R_{\vec{u}} \cdot M_{\vec{v}} \subseteq M_{\vec{u}+\vec{v}}$.

Def. X is multifiltered if we are given a family of subspaces $\{X_{\vec{v}} \subseteq X\}_{\vec{v} \in \mathbb{N}}$ with inclusions $X_{\vec{u}} \subseteq X_{\vec{w}}$ whenever $\vec{u} \lesssim \vec{w}$ s.t. the diagrams commute.

$$\begin{array}{ccc} X_{\vec{u}} & \longrightarrow & X_{\vec{v}_1} \\ \downarrow & & \downarrow \\ X_{\vec{v}_2} & \longrightarrow & X_{\vec{w}} \end{array} \quad \left(\forall\ \vec{u} \lesssim \begin{matrix}\vec{v}_1 \\ \vec{v}_2\end{matrix} \lesssim \vec{w}\ .\right)$$

Def. A persistence module M is a family of $\mathbb{k}$-modules $\{M_{\vec{v}}\}_{\vec{v}}$ together with homomorphisms $\varphi_{\vec{u},\vec{v}} : M_{\vec{u}} \to M_{\vec{v}}$ for all $\vec{u} \lesssim \vec{v}$ s.t.

$\varphi_{\vec{u},\vec{v}} \circ \varphi_{\vec{v},\vec{w}} = \varphi_{\vec{u},\vec{w}}$ whenever $\vec{u} \lesssim \vec{v} \lesssim \vec{w}$.

Def. Given a persistence module M, we define an n-graded module over A_n by

$$\alpha(M) = \bigoplus_{v} M_v$$

where the $\mathbb{k}$-module structure is the direct sum structure, and we require that $x^{\vec{v}-\vec{u}} : M_{\vec{u}} \to M_{\vec{v}}$ is $\varphi_{u,v}$ whenever $\vec{u} \lesssim \vec{v}$

Correspondence:

Thm. the category of multidimensional persistence modules f.t.

$$\Updownarrow$$

the category of n-graded modules over A_n f.t.

Classification.

Def. n-graded set (X, φ) where X is a set and $\varphi : X \to \mathbb{Z}^n$.

the map f of n-graded sets :
$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[r, "\varphi"] & \mathbb{Z}^n \\ Y \arrow[ur, "\psi"'] & \end{tikzcd}$$

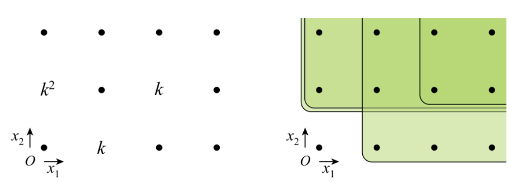
For any n-graded module M over A_n $\mathbb{H} =: \bigcup_{v \in \mathbb{Z}^n} M_v$

A free A_n-module on the graded set (X, φ) F

for any n-graded A_n-module M and map of n-graded sets $\theta : (X, \varphi) \to \mathbb{H}(M)$, there is a unique homomorphism $\lambda : F \to M$ of n-graded A_n-modules so that the diagram

$$\begin{tikzcd} (X, \varphi) \arrow[r, "\eta"] \arrow[dr, "\theta"'] & \mathbb{H}(F) \arrow[d, "\mathbb{H}(\lambda)"] \\ & \mathbb{H}(M) \end{tikzcd}$$

commutes



Def. the type of an n-graded vector space V is the unique multiset which is isomorphic to a graded set basis for V, denote it $\xi(V)$.

Similarly, we can define $\xi(F)$ for free object F.

For any n-graded vector space V, we have a free n-graded module $F(V)$, s.t. $\Bbbk \otimes_{A_n} F(V) \cong V$

For any multiset ξ of $\mathbb{N}^n$, we can also consider $V(\xi)$ and $F(\xi)$

For any n-graded module M, consider the minimal free solution of M

$$\cdots \to F_1 \to F_0 \to M \to 0$$

$\xi_0(M) =: \xi(F_0)$ $\quad \xi_1(M) =: \xi(F_1)$

$S(F, \xi) =: \{ L \mid L \text{ is a } A_n\text{-submodule and } \xi(L) = \xi_1 \}$

$I(\xi_0, \xi_1) =: \{ [M] \text{ (isomorphism class)} \mid \xi_0(M) = \xi_0, \xi_1(M) = \xi_1 \}$

We have map $q: S(F_0, \xi_1) \to I(\xi_0, \xi_1)$

$$L \mapsto [F_0/L]$$

Thm. bijection: $\{ \text{the orbits of } \mathrm{Aut}(F_0) \curvearrowright S(F_0, \xi_1) \} \xrightarrow{1\text{-}1} I(\xi_0, \xi_1)$

Parameterization (for $S(F, \xi_1)$)

The goal is to demonstrate that $ARR_{\xi,\delta}(F)$ is in bijective correspondence with the set of points of a quasi-projective variety over the field $\Bbbk$

① $\xi = (V, \alpha)$ $\quad V \subseteq \mathbb{N}^n$ $\quad \alpha: V \to \mathbb{N}$ $\quad$ i.e. ξ is a multiset

$\delta: V \to \mathbb{Z}$ be any function.

$ARR_{\xi,\delta}(F) = \{ \text{assignments } v \to L_v \mid v \in V \quad L_v \text{ is a } \Bbbk\text{-linear subspace}$

of F_v satisfying the three conditions :

1. $\vec{v}' \leq \vec{v} \Rightarrow x^{\vec{v}-\vec{v}'} L_{\vec{v}'} \subseteq L_{\vec{v}}$
2. $\dim_{\mathbb{k}}(L_{\vec{v}}) = \delta(\vec{v})$
3. $\dim_{\mathbb{k}}(L_{\vec{v}} / \sum_{v' < v} x^{\vec{v}-\vec{v}'} L_{\vec{v}'}) = \alpha(\vec{v})$ for all $\vec{v} \in V$.

(depict submodule L of M)

$$\Rightarrow ARR_{\xi,\delta}(F) \subseteq \mathcal{E} = \prod_{\vec{v} \in V} Gr_{\delta(\vec{v})}(F_{\vec{v}})$$

② $F = \bigoplus_{\vec{v}} A_n(\vec{v})$ $\xi = (V, \alpha)$

Choose a "large" $\vec{v}^*$, s.t. $\vec{v}^* \gtrsim \vec{v}$

$$\begin{array}{ccc} F_{\vec{v}} \stackrel{\wedge}{=} x^{\vec{v}^*-\vec{v}} \cdot F_{\vec{v}} & \subseteq & F_{\vec{v}^*} \\ \cup\!\text{l} & & \cup\!\text{l} \\ L_{\vec{v}} \stackrel{\wedge}{=} L^*_{\vec{v}} & & \end{array}$$

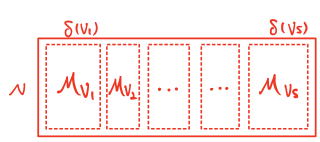
Condition : 1. $L^*_{\vec{v}} \subseteq x^{\vec{v}^*-\vec{v}} F_{\vec{v}}$

2. If $\vec{v} \lesssim \vec{v}'$, then $L^*_{\vec{v}} \subseteq L^*_{\vec{v}'}$

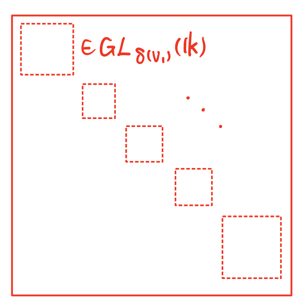
3. $\dim_{\mathbb{k}}(L^*_{\vec{v}}) = \delta(v)$

4. $\dim_{\mathbb{k}}(L^*_{\vec{v}} / \sum_{v' \lnsim v} L^*_{\vec{v}'}) = \alpha(v)$ for any $v \in V$.

③ $N = \dim(F_{v^*})$



⟷ a (ξ, δ)-frame



1. $L_v^* \subseteq G_v$ for all v: This condition is already accounted for with the requirement that the blocks of M_v corresponding to the set $B(v')$ is zero if V.
2. If $v \leq v'$, then $L_v^* \subseteq L_{v'}^*$: This condition can be reinterpreted as the requirement that the $N \times (\delta(v) + \delta(v'))$ matrix

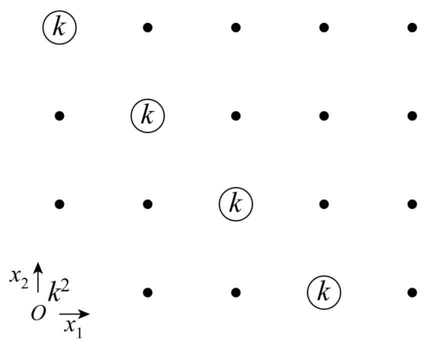
$$\mu(v, v') = \left[M_v \mid M_{v'}\right]$$

   has rank $\delta(v')$, or equivalently that all its $(\delta(v') + 1) \times (\delta(v') + 1)$ minors vanish. This is clearly an algebraic condition, invariant under the group action.
3. $\dim_k(L_v^*) = \delta(v)$: This condition is already accounted for in the injectivity condition defining the variety of frames.
4. $\dim_k(L_v^* / \sum_{v'<v} L_{v'}^*) = \alpha(v)$ for all $v \in V$: This condition can be reinterpreted as the requirement that the $N \times (\sum_{v'<v} \delta(v'))$ matrix

$$\lambda(v) = \left[M_{v_1'} \mid M_{v_2'} \mid \ldots \mid M_{v_j'}\right],$$

where $\{v_1', \ldots, v_j'\}$ is an enumeration of all v_i's for which $v_i < v$, has rank exactly $\delta(v) - \alpha(v)$. This means that this set can be obtained as the action invariant Zariski closed set for which all the $(\delta(v) - \alpha(v) + 1) \times (\delta(v) - \alpha(v) + 1)$ minors of $\lambda(v)$ vanish, and removing from it the invariant closed Zariski closed set for which all the $(\delta(v) - \alpha(v)) \times (\delta(v) - \alpha(v))$ minors vanish.

Example. $\xi_0=\{(0,0),2\}$ $\quad \xi_1=\{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}$



$$\mathrm{GL}\big(F(\xi_0)\big)=\mathrm{GL}\big(k^2\big)=\mathrm{GL}_2(k),$$

$$\begin{array}{ccccc}
GL(F(\xi_0)) & \curvearrowright & S(F_0,\xi_1) & \text{orbits} \leftrightsquigarrow & I(\xi_0,\xi_1) \\
\| & & \| & & \\
GL_2(\mathbb{k}) & & \prod_v Gr_{\dim F(\xi_1)_v}(F(\xi_0)_v) & & \\
 & & \| & & \\
 & & \prod_v Gr_1(\mathbb{k}^2) = \mathbb{P}^1(\mathbb{k})^4 & &
\end{array}$$

$\Rightarrow \quad GL_2(\mathbb{k}) \curvearrowright \mathbb{P}^1(\mathbb{k})^4$

Let $l_i \in \mathbb{P}^1(\mathbb{k})$, then $(l_1,l_2,l_3,l_4)\in \mathbb{P}^1(\mathbb{k})^4$

We consider the subset $\Omega \subseteq \mathbb{P}^1(\mathbb{k})^4$ $\quad \Omega=\{(l_1,l_2,l_3,l_4) \mid l_i\neq l_j \ i\neq j\}$

$\Rightarrow \quad (l_1,l_2,l_3,l_4) \Rightarrow (x\text{-axis},\ y\text{-axis},\ \{x=y\},\ l)$

$\Rightarrow \Omega \overset{1-1}{\longleftrightarrow} \mathbb{P}^1-\{0,\infty,1\} = \mathbb{k}-\{0,1\}.$