Interleaving distance

Def. C = cat. $(\mathbb{R}, \geq)$. partially ordered cat.

A persistent object is a functor $F: \mathbb{R} \to C$. $(\mathbb{R}^{op} \to C)$

Def. $F: \mathbb{R} \to C$. $\varepsilon \in \mathbb{R}$. A ε-shifting of F is another

persistent obj. $F[\varepsilon]: \mathbb{R} \to C$, $F[\varepsilon]_r := F_{r+\varepsilon}$. $\forall r \in \mathbb{R}$.

(if $F: \mathbb{R}^{op} \to C$, $F[\varepsilon]^r := F^{r-\varepsilon}$.)

Rmk: $\exists$ natural transformation $F \to F[\varepsilon]$

$$F_r \to F_{r+\varepsilon}$$

Def. $F, G : \mathbb{R} \to C$. An ε-morphism from F to G is a natural transformation $F \to G[\varepsilon]$

Def. $F, G : \mathbb{R} \to C$ ($\mathbb{R}^{op} \to C$). Call F, G ε-interleaved if $\exists$ two ε-morphisms $\mu : F \to G[\varepsilon]$, $\nu : G \to F[\varepsilon]$ s.t.

$$\begin{tikzcd}
 & F \arrow[rr] \arrow[dr, "\mu"] & & F[2\varepsilon] \arrow[dr, "\mu[2\varepsilon]"] & \\
G[-\varepsilon] \arrow[ur, "\nu[-\varepsilon]"] \arrow[rr] & & G[\varepsilon] \arrow[ur, "\nu[\varepsilon]"] \arrow[rr] & & G[3\varepsilon]
\end{tikzcd}$$

commutes

The interleaving distance of F, G is $d_C(F, G) :=$
$\inf\{\varepsilon \geq 0 \mid F, G \ \varepsilon\text{-interleaved}\}$

Rmk: If we have $\underbrace{H: C \to D}$ $d_D(HF, HG) \leq d_C(F, G)$

Thm. (Morse-type stability) X. top space. $f, g: X \to \mathbb{R}$ cts. Consider $X^f: \mathbb{R} \to \mathrm{Top}$, $r \mapsto f^{-1}(-\infty, r]$

$X^g: \mathbb{R} \to \mathrm{Top}$, $r \mapsto g^{-1}(-\infty, r]$. Then $d_{\mathrm{Top}}(X^f, X^g) \leq \|f-g\|_\infty$

$\|f-g\|_\infty := \sup_{x \in X} \|f(x) - g(x)\|$

Pf: If $\|f-g\|_\infty = \infty$. $\checkmark$ Otherwise. $\forall \varepsilon \geq \|f-g\|_\infty$.

$$\begin{tikzcd}
X^f_r \arrow[rr] \arrow[dr] & & X^f_{r+2\varepsilon} \arrow[dr] & \\
& X^g_{r+\varepsilon} \arrow[ur] \arrow[rr] & & X^g_{r+3\varepsilon}
\end{tikzcd}$$

ε-interleaving

$$\|f-g\|_\infty \le \varepsilon \implies f^{-1}(-\infty, r] \subseteq g^{-1}(-\infty, r+\varepsilon] \subseteq f^{-1}(-\infty, r+2\varepsilon] \subseteq g^{-1}(-\infty, r+3\varepsilon] \quad \square$$

$R = k$.

Def. $X, Y : \mathbb{R} \to \mathrm{Top}$. Define $d_{\mathrm{Vect}}(X, Y) := d_{\mathrm{Vect}}(H^*(X), H^*(Y))$

(Vect : graded vector space)

Def. X. top. space. $C^*(X)$: singular cochains. There is a product $- \cup - : C^i(X) \times C^j(X) \to C^{i+j}(X)$

$$(f \cup g)(\sigma) := f(\sigma|_{[v_0, \dots, v_i]})\, g(\sigma|_{[v_i, \dots, v_{i+j}]})$$

$\sigma : \Delta^{i+j} \to X$. For $[f], [g] \in H^*(X)$, can lift to $C^*(X)$

and define $[f] \cup [g]$.

Def. A differential graded associative algebra is a graded algebra A with a differential map $\delta : A \to A$ of degree 1 s.t.

(1) $\delta \circ \delta = 0$

(2) $\delta(a \cdot b) = \delta a \cdot b + (-1)^{\deg(a)} a \cdot \delta b$

Denote the category of d.g. associative alg by Alg_{dg}.

$C^*(X)$, $H^*(X) \in Alg_{dg}$.

Both Ch_k (cochain complexes over k), and dg-Alg have natural notion of homotopy.

$\Rightarrow ho(Ch_k)$. $ho(Alg_{dg})$.

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & Top^{op} \arrow[dr, "C_{ho(Alg_{dg})}"] \arrow[drr, bend left, "C_{ho(Ch_k)}"] \arrow[ddr, bend right, "H^*_{Alg_{dg}}"'] & & \\
& & \underline{ho(Alg_{dg})} \arrow[r, "forget"] \arrow[d, "H"] & \underline{ho(Ch_k)} \arrow[d, "H"] \\
& & Alg_{dg} \arrow[r, "forget"] & Vect
\end{tikzcd}
$$

Notation: $X, Y : \mathbb{R} \to \mathrm{Top}$. Define the dg-algebra interleaving distance of X, Y as $d_{dg}(X,Y) :=$

$$d_{\mathrm{Alg}_{dg}}(H^*(X), H^*(Y))$$

$$\begin{tikzcd} A_1 \arrow[r] \arrow[dr, "H"] & A_2 \arrow[r] \arrow[dr, "H"] & A_3 \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \end{tikzcd}$$

?

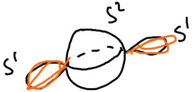
Prop. For any $X, Y : \mathbb{R} \to \mathrm{Top}$, we have:

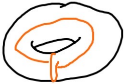
$$d_{\mathrm{vert}}(X,Y) \le d_{\mathrm{Alg}_{dg}}(X,Y) \le d_{\mathrm{ho}(\mathrm{Alg}_{dg})}(C^*(X), C^*(Y))$$

$$d_{\mathrm{vert}}(X,Y) \le d_{\mathrm{ho}(\mathrm{Ch}_k)}(C^*(X), C^*(Y)) \le d_{\mathrm{ho}(\mathrm{Alg}_{dg})}(C^*(X), C^*(Y)).$$

$d_{\mathrm{vert}}(H^*(X), H^*(Y))$

Ex.





$$X = \{(x,y,0) \mid x^2+(y-2)^2=1\} \cup \{x^2+y^2+z^2=1\} \cup \{(x,y,0) \mid x^2+(y+2)^2=1\}$$

$$Y = \{(x,y,z) \mid (x^2+y^2+z^2+3)^2 = 16(x^2+y^2)\}$$

We have $H^0(X)=k$, $H^1(X)=k^2$, $H^2(X)=k$.

$H^0(Y)=k$, $H^1(Y)=k^2$, $H^2(Y)=k$.

As graded vector spaces, $H^*(X) \cong H^*(Y)$.

Fix $\varepsilon \geq 0$ small

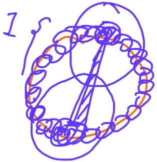
$$X_\varepsilon := \bigcup_{x\in X} B(x,\varepsilon) \qquad Y_\varepsilon := \bigcup_{y\in Y} B(y,\varepsilon)$$

when $\varepsilon < 1$, we have $X_\varepsilon \simeq X$, $Y_\varepsilon \simeq Y$.

Fix $0<\alpha<<\frac{1}{2}$. $\exists$ finitely many points $\{x_i\}_{i\in I} := D_X^\alpha \subseteq \underline{X_\alpha}$

s.t. $\{B(x_i,\alpha)\}_{i\in I}$ covers X_α. Similar for Y.

Consider Čech complex $\check{C}(D_X^\alpha)_*$, $\check{C}(D_Y^\alpha)_* : \mathbb{R} \to \Delta\mathrm{Cpx}$.



For $\alpha < r < \underline{1-\alpha}$, we have $\check{C}(D_X^\alpha)_r \simeq X_\alpha \simeq X$.

When $r > 1-\alpha$, $\check{C}(D_X^\alpha)_r$, $\check{C}(D_Y^\alpha)$ are contractible.

Prop. Keeping the notions, we have

$$d_{\mathrm{Vect}}(\check{C}(D_X^\alpha)_*, \check{C}(D_Y^\alpha)_*) \le 2\alpha$$

$$\frac{1-2\alpha}{2} \le d_{\mathrm{Alg}_{dg}}(\check{C}(D_X^\alpha)_*, \check{C}(D_Y^\alpha)_*) \le \frac{1}{2}$$

$$\begin{tikzcd} F \arrow[r, "0"] \arrow[d, "0"'] & F[1] \\ G[\tfrac{1}{2}] \arrow[ur, "0"'] & \end{tikzcd}$$

Pf: When $r \in (\alpha, 1-\alpha)$, $\check{C}(D_X^\alpha)_r \simeq X_\alpha \simeq X$, $\check{C}(D_Y^\alpha) \simeq Y_\alpha \simeq Y$

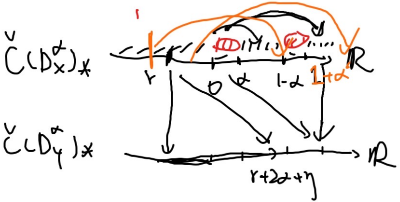
$H^*(X) \cong H^*(Y)$ (in Vect), $\Rightarrow H^*(\check{C}(D_X^\alpha)_r) \cong H^*(\check{C}(D_X^\alpha)_r)$

when $r > 1+\alpha$, both $\check{C}(D_X^\alpha)_r$, $\check{C}(D_Y^\alpha)_r$ contractible

$H^*(\check{C}(D_X^\alpha)_r) \cong H^*(\check{C}(D_Y^\alpha)_r) = 0$

$\forall \eta > 0$, claim $\exists (2\alpha+\eta)$-interleaving

$\check{C}(D_X^\alpha)_r \longrightarrow \check{C}(D_Y^\alpha)_{r+2\alpha+\eta}$



By contradiction, if $\exists\ \varepsilon < \frac{1-2\alpha}{2}$ s.t. there is an ε-interleaving from $H^*(\check{C}(D_X^\alpha)_*)$ to $H^*(\check{C}(D_Y^\alpha)_*)$

$\exists$ commutative diagram

$1-\alpha \quad \alpha$

$$\begin{tikzcd}
 & H^*(\check{C}(D_X^\alpha)_{\alpha+\varepsilon}) = H^*(X) \arrow[dr, "\nu"] & \\
H^*(\check{C}(D_Y^\alpha))_\alpha \arrow[ur, "\mu"] \arrow[rr, "\cong"] & & H^*(\check{C}(D_Y^\alpha)_{\alpha+2\varepsilon})
\end{tikzcd}$$

$$H^*(\check{C}(D_Y^\alpha))_\alpha = H^*(Y)$$

$\alpha+\varepsilon < \alpha+2\varepsilon < 1-\alpha$

$\Rightarrow \check{C}(D_X^\alpha)_{\alpha+\varepsilon} \simeq X_{\alpha+\varepsilon} \simeq X$
$\simeq \check{C}(D_X^\alpha)_{\alpha+2\varepsilon}$.

$\check{C}(D_Y^\alpha)_{\alpha+\varepsilon} \simeq \check{C}(D_Y^\alpha)_{\alpha+2\varepsilon} \simeq Y$

$\Rightarrow \mu$ injective

$\Rightarrow \exists\ H^*(Y) \hookrightarrow H^*(X)$.

contradiction. $\square$

Massey Product & A_∞ - algebra

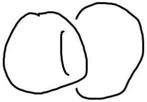
Def. (Triple Massey product) X : top space.

k : field. For $x \in C^i(X,k)$, denote $\bar{x} := (-1)^{1+i} x$.

For $x_1 \in C^i(X,k)$, $x_2 \in C^j(X,k)$, $x_3 \in C^s(X,k)$

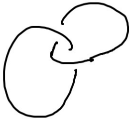
The Massey triple product $\langle [x_1], [x_2], [x_3] \rangle$ is the subset $\{ w := \bar{x}_{12} \cup x_3 + \bar{x}_1 \cup x_{23} \mid \delta x_{12} = \bar{x}_1 \cup x_2,\ \delta x_{23} = \bar{x}_2 \cup x_3 \}$ of $H^{i+j+s-1}(X)$.

Rmk: $\langle [x_1], [x_2], [x_3] \rangle$ is nonempty if $[x_1] \cup [x_2] = 0$.
$[x_2] \cup [x_3] = 0$.



T

$\mathbb{R}^3 - T$

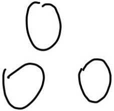


H

$\mathbb{R}^3 - H$

Borromean rings

$\mathbb{R}^3 - B$



U

$\mathbb{R}^3 - U$

Def. An A_∞-algebra is a graded vector space $A = \{A_k\}_{k\in\mathbb{Z}}$, equipped n-ary operations $m_n : A^{\otimes n} \to A$ of degree $n-2$, $\forall n \geq 1$ s.t.

$$\sum_{p+q+r=n} \underline{(-1)^{p+qr} m_{p+r+1} \circ (id_A^{\otimes p} \otimes m_q \otimes id_A^{\otimes r})} = 0, \quad \forall n \geq 1.$$

Rmk: (1) $n=1$, we have $m_1 \circ m_1 = 0$

$n=2$, the equation is the Leibnitz's rule. δ

(2) A differential graded algebra is an A_∞-algebra with $m_n = 0$, $\forall n \geq 3$

$H^*(X)$ with Massey products is an A_∞-algebra.

$$
\begin{tikzcd}
\mathrm{Top} \arrow[dr, "C^*_{A_\infty}"] \arrow[drr, bend left, "C^*_{dg\text{-}Alg}"] \arrow[drrr, bend left, "C^*_{Ch_k}"] \arrow[ddr, bend right, "H^*_{A_\infty}"'] & & & \\
& A_\infty\text{-alg} \arrow[r, "f"] \arrow[d, "H^*"] & dg\text{-}Alg \arrow[r, "f"] \arrow[d, "H^*"] & Ch_k \arrow[d, "H^*"] \\
& A_\infty\text{-alg} \arrow[r, "f"] & dg\text{-}Alg \arrow[r, "f"] & Vect
\end{tikzcd}
$$

$$\underline{d_{A_\infty\text{-}Alg}(X,Y)} \geq \underline{d_{dg\text{-}Alg}(X,Y)} \geq d_{Vect}(X,Y)$$

How blind is

Stability

Def. X, Y be two sets.

(1) A multi-valued map from X to Y is a subset C of $X \times Y$ s.t. $\pi_X|_C : C \to X$ is surjective, where $\pi_X : X \times Y \to X$. Denote by $C : X \rightrightarrows Y$

(2) The image of $S \subseteq X$ under C is $C(S) := \pi_Y(\pi_X^{-1}(S) \cap C)$

(3) A map $f : X \to Y$ is called subordinate to C if $\forall x \in X$, $(x, f(x)) \in C$, write $f : X \xrightarrow{C} Y$

(4) Given $C : X \rightrightarrows Y$. if $\pi_Y|_C : C \to Y$ surjective

Then call C a correspondence. Define C^{T} the transpose of C. $C^{T} := \{(y,x) \in Y \times X \mid (x,y) \in C\}$

Def. (Gromov-Hausdorff distance). (X, d_X), (Y, d_Y) metric spaces. $C : X \rightrightarrows Y$ a correspondence. The distorsion of C is

$$\mathrm{dist}(C) := \sup_{(x,y),(x',y') \in C} |d_X(x,x') - d_Y(y,y')|$$

The G-H distance is defined to be

$$d_{GH}(X,Y) := \frac{1}{2} \inf_{C : X \rightrightarrows Y} \mathrm{dist}(C)$$

w/. $I = \{(\eta : X \to Z, \tau : Y \to Z) \mid (Z, d_Z)$ metric space, η, τ iso. emb.$\}$

Rmk: $d_{GH}(X,Y) = \inf_{(\eta,\tau) \in I} \min\{\varepsilon \geq 0 \mid \eta(X) \subseteq \tau(Y)_\varepsilon,\ \tau(Y) \subseteq \eta(X)_\varepsilon\}$

$$\tau(Y)_\varepsilon := \bigcup_{y \in \tau(Y)} B_Z(y, \varepsilon)$$

Theorem. (X, d_X), (Y, d_Y) metric spaces.

Then $\underline{d_{\mathrm{Alg}_{dg}}}(H^*(R(X)), H^*(R(Y))) \leq \underline{d_{GH}(X, Y)}$

Rips complex

$d_{\mathrm{Alg}_{dg}}(H^*(\check{C}(X)), H^*(\check{C}(Y))) \leq d_{GH}(X, Y)$

Lemma. $d_{A_\infty}(X, Y) = d_{\mathrm{ho}(\mathrm{Alg}_{dg})}(C^*(X), C^*(Y))$

$\underline{C : X \rightrightarrows Y}$ $\qquad R(X)_r \to R(Y)_{r+\varepsilon}$

$\check{C}(X)_r \to \check{C}(Y)_{r+\varepsilon}$